

# ON THE SQUARE FUNCTION ASSOCIATED WITH GENERALIZED BOCHNER-RIESZ MEANS

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**ABSTRACT.** We consider generalized Bochner-Riesz multipliers of the form  $(1 - \rho(\xi))_+^\lambda$  where  $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}$  belongs to a class of rough distance functions homogeneous with respect to a nonisotropic dilation group. We prove a critical  $L^4$  estimate for the associated square function, which we use to derive multiplier theorems for multipliers of the form  $m \circ \rho$  where  $m : \mathbb{R} \rightarrow \mathbb{C}$ .

## 1. INTRODUCTION

The characterization of Fourier multiplier operators that are bounded on  $L^p$  when  $p \neq 1, 2$  is a difficult open problem that has a long and rich history in harmonic analysis. A particular special case that has been especially studied is the class of radial Fourier multipliers, for which the Bochner-Riesz multipliers are prototypical examples. In [3], Carbery, Gasper and Trebels proved sufficient conditions for a radial function on  $\mathbb{R}^2$  to be a Fourier multiplier on  $L^p(\mathbb{R}^2)$ . Their theorem can be stated as follows.

**Theorem A** ([3]). *Let  $m : (0, \infty) \rightarrow \mathbb{C}$  be bounded and measurable. Then for  $4/3 \leq p \leq 4$  and  $\alpha > 1/2$ ,*

$$\|m(|\cdot|)\|_{M^p(\mathbb{R}^2)} \lesssim \sup_{t>0} \left( \int |\mathcal{F}_{\mathbb{R}}[\phi(\cdot)m(t\cdot)](\tau)|^2 |\tau|^{2\alpha} d\tau \right)^{1/2}.$$

Theorem A is sharp, as can be verified by comparing with the known sharp  $L^p$  bounds for Bochner-Riesz multipliers in  $\mathbb{R}^2$  (see [7]). Theorem A was obtained as a consequence of a critical  $L^4$  estimate for the Bochner-Riesz square function in  $\mathbb{R}^2$ , proved by Carbery in [2].

In this paper, we extend the result of Theorem A to a class of *quasiradial* multipliers of the form  $m \circ \rho$ , where  $\rho$  belongs to a class of rough distance functions homogeneous with respect to a *nonisotropic* dilation group. Here we may view  $\rho(\xi)$  as generalizing the function  $|\xi|$ , which corresponds to

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the special case of radial multipliers. Our consideration of such a class of distance functions is in part motivated by the work of Seeger and Ziesler in [19], where the authors consider Bochner-Riesz means of the form  $(1 - \rho(\xi))_+^\lambda$  where  $\rho$  is the Minkowski functional of a bounded convex domain in  $\mathbb{R}^2$  containing the origin. However, the class of distance functions we work with is more general than what is considered in [19], since it also includes distance functions  $\rho$  that have nonisotropic homogeneity.

As motivated by [19], let  $\Omega \subset \mathbb{R}^2$  be a bounded, open convex set containing the origin. Since the results in this paper are dilation invariant, we will assume that  $\Omega$  contains the ball of radius 8 centered at the origin. Let  $M > 0$  be the smallest positive integer such that

$$(1.1) \quad \{\xi : |\xi| \leq 8\} \subset \Omega \subset \overline{\Omega} \subset \{\xi : |\xi| < 2^M\}.$$

This quantity  $M$  associated to such a convex domain  $\Omega$  is an important parameter on which our results will depend. One may note that it determines the Lipschitz norm of parametrizations of  $\partial\Omega$ .

We now introduce the notion of a nonisotropic dilation group. Let  $A$  be a  $2 \times 2$  matrix with eigenvalues  $\lambda_1, \lambda_2$  (not necessarily distinct) such that  $\operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2) > 0$ . A *nonisotropic dilation group* associated to  $A$  is a one-parameter family  $\{t^A : t > 0\}$ , where  $t^A = \exp(\log(t)A)$ . We say that a pair  $(\Omega, A)$  is *compatible* if it satisfies the following:

- (1) For any  $\xi \in \mathbb{R}^2 \setminus \{0\}$  the orbit  $\{t^A \xi : t > 0\}$  intersects  $\partial\Omega$  exactly once,
- (2) If  $\Theta(\Omega, A)$  denotes the infimum of all angles between the tangent vector to an orbit  $\{t^A \xi : t > 0\}$  at  $\xi$  and a supporting line at  $\xi$  for any  $\xi \in \partial\Omega$ , then  $\Theta(\Omega, A) > 0$ .

We associate to a compatible pair  $(\Omega, A)$  a norm function  $\rho \in C(\mathbb{R}^2)$ , defined by setting  $\rho(0) = 0$  and setting  $\rho(\xi)$  to be the unique  $t$  such that  $t^{-A}\xi \in \partial\Omega$  if  $\xi \neq 0$ . If  $\partial\Omega$  is smooth, then  $\rho \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ . To see this, apply the implicit function theorem to  $F(x, t) = \operatorname{dist}(t^A x, \partial\Omega)$ . Moreover, we also have  $\|\rho\|_{C^{0,1}(K)} \lesssim_{K, M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)} 1$  for any compact  $K \subset \mathbb{R}^2 \setminus \{0\}$ .

Note that in the special case that  $A$  is the identity,  $(\Omega, A)$  is a compatible pair for any bounded, open convex set  $\Omega$  satisfying (1.1), and we have  $\Theta(\Omega, A) \gtrsim_M 1$ . It was noted in [23] that for every  $A$  there exists a compatible pair  $(\Omega, A)$  obtained by taking  $\Omega$  to be the region bounded by  $\{\xi \in \mathbb{R}^2 : \langle B\xi, \xi \rangle = 1\}$ , where  $B$  is the positive definite symmetric matrix given by

$$B = \int_0^\infty \exp(-tA^*) \exp(-tA) dt.$$

See [23] for more details. In this particular case  $\partial\Omega$  is smooth; however as already noted in this paper we consider general convex domains, with special emphasis on the case when  $\partial\Omega$  is rough.

**Notation.** Throughout the rest of the paper, in every situation where it is clear that we have fixed a compatible pair  $(\Omega, A)$ , we will write  $\lesssim$ ,  $\gtrsim$  and  $\approx$  to denote inequalities where the implied constant possibly depends on  $M$ ,  $\text{Re}(\lambda_1)$ ,  $\text{Re}(\lambda_2)$ , and  $\Theta(\Omega, A)$ . We will also assume that all explicit constants that appear possibly depend on  $M$ ,  $\text{Re}(\lambda_1)$ ,  $\text{Re}(\lambda_2)$ , and  $\Theta(\Omega, A)$ .

Given a compatible pair  $(\Omega, A)$ , define the Bochner-Riesz means  $R_t^\lambda f$  associated with  $(\Omega, A)$  for Schwartz functions  $f \in \mathcal{S}(\mathbb{R}^2)$  by

$$\mathcal{R}_t^\lambda f(x) = \frac{1}{(2\pi)^2} \int_{|\xi| \leq t} \left(1 - \frac{\rho(\xi)}{t}\right)^\lambda \hat{f}(\xi) e^{i\langle \xi, x \rangle} d\xi.$$

Define the Bochner-Riesz square function  $G^\lambda f$  associated with  $(\Omega, A)$  for Schwartz functions  $f \in \mathcal{S}(\mathbb{R}^2)$  by

$$G^\lambda f(x) = \left( \int_0^\infty |\mathcal{R}_t^\lambda f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Our main result is the following critical  $L^4$  estimate for the Bochner-Riesz square function.

**Theorem 1.1.** *Let  $(\Omega, A)$  be a compatible pair, and let  $G^\lambda f$  denote the Bochner-Riesz square function associated to  $(\Omega, A)$ . For  $\lambda > -1/2$ ,*

$$\left\| G^\lambda f \right\|_{L^4(\mathbb{R}^2)} \lesssim \|f\|_{L^4(\mathbb{R}^2)}$$

for  $f \in \mathcal{S}(\mathbb{R}^2)$ .

Following [3], we obtain the subsequent corollary, which is an extension of the result of Theorem A to quasiradial multipliers of the form  $m \circ \rho$ .

**Corollary 1.2.** *Let  $(\Omega, A)$  be a compatible pair with associated norm function  $\rho$ . Let  $m : \mathbb{R} \rightarrow \mathbb{C}$  be measurable function with  $\|m\|_{L^\infty(\mathbb{R})} \leq 1$ . Then for  $4/3 \leq p \leq 4$  and  $\alpha > 1/2$ ,*

$$\|m \circ \rho\|_{M^p(\mathbb{R}^2)} \lesssim \sup_{t>0} \left( \int |\mathcal{F}_\mathbb{R}[\phi(\cdot)m(t\cdot)](\tau)|^2 |\tau|^{2\alpha} d\tau \right)^{1/2}.$$

To prove Theorem 1.1, we will first decompose the multiplier  $(1 - \rho(\xi))_+^\lambda$  in a standard fashion into smooth functions supported on “annuli” of thickness comparable to the distance from  $\partial\Omega$  (for example, see [4], [2]). Theorem 1.1 then reduces to proving the following proposition.

**Proposition 1.3.** *Let  $(\Omega, A)$  be a compatible pair. Fix a Schwartz function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  supported in  $[-1, 1]$  with  $|\Phi| \leq 1$ . There is a constant  $C > 0$  such that for every  $\epsilon > 0$  and every  $0 < \delta < C$ ,*

$$\left\| \left( \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \lesssim_\epsilon \delta^{1/2-\epsilon} \|f\|_4,$$

where

$$\psi_t(x) = \mathcal{F}(\phi(\frac{\rho(\cdot)}{t}))(x), \quad \phi(\xi) = \Phi(\frac{\xi-1}{\delta}).$$

The overall structure of the proof of Proposition 1.3 will follow [2] and [19], and will draw heavily on the techniques therein. However, the presence of nonisotropic dilations and the roughness of  $\partial\Omega$  introduces new difficulties to the proof since the underlying geometry becomes more complicated, requiring more intricate decompositions on the Fourier side as well as a more sophisticated use of Littlewood-Paley inequalities.

## 2. PRELIMINARIES ON CONVEX DOMAINS IN $\mathbb{R}^2$

*Elementary facts about convex functions in  $\mathbb{R}^2$ .* We note for later use the following lemma, which can be found in [19]. The proof is straightforward and we omit it here, and the reader is encouraged to refer to [19] for a proof.

**Lemma 2.1** ([19]).  $\partial\Omega \cap \{x : -1 \leq x_1 \leq 1\}$  can be parametrized by

$$(2.1) \quad t \mapsto (t, \gamma(t)), \quad -1 \leq t \leq 1,$$

where

(1)

$$(2.2) \quad 1 < \gamma(t) < 2^M, \quad -1 \leq t \leq 1.$$

(2)  $\gamma$  is a convex function on  $[-1, 1]$ , so that the left and right derivatives  $\gamma'_L$  and  $\gamma'_R$  exist everywhere in  $(-1, 1)$  and

$$(2.3) \quad -2^{M-1} \leq \gamma'_R(t) \leq \gamma'_L(t) \leq 2^{M-1}$$

for  $t \in [-1, 1]$ . The functions  $\gamma'_L$  and  $\gamma'_R$  are decreasing functions;  $\gamma'_L$  and  $\gamma'_R$  are right continuous in  $[-1, 1]$ .

(3) Let  $\ell$  be a supporting line through  $\xi \in \partial\Omega$  and let  $n$  be an outward normal vector. Then

$$(2.4) \quad |\langle \xi, n \rangle| \geq 2^{-M} |\xi|.$$

*Reduction to the case when  $\partial\Omega$  is smooth.* Motivated by [19], Lemma 2.2, we will show that it suffices to prove Proposition 1.3 with the implied constant depending only on  $M$  (and not, for instance, the  $C^2$  norm of local parametrizations of  $\partial\Omega$ ) in the special case that  $\partial\Omega$  is smooth. The first step is to approximate  $\Omega$  by a sequence of convex domains with smooth boundaries satisfying the same quantitative estimates as  $\Omega$ .

**Lemma 2.2.** *Let  $(\Omega, A)$  be a compatible pair. There is a sequence of convex domains  $\{\Omega_n\}$  satisfying the following:*

- (1)  $\partial\Omega_n$  is  $C^\infty$ ,
- (2) For  $n$  sufficiently large,  $(\Omega_n, A)$  is a compatible pair and  $\Theta(\Omega_n, A) \geq \Theta(\Omega, A)/2$ ,
- (3) For each  $n$  we have
$$\{\xi : |\xi| \leq 4\} \subset \Omega_n \subset \overline{\Omega_n} \subset \{\xi : |\xi| < 2^{M+1}\},$$
- (4)  $\lim_{n \rightarrow \infty} \rho_n(\xi) = \rho(\xi)$  with uniform convergence on compact sets.

*Proof.* We adopt the same approach as in [19], namely, approximating  $\Omega$  by convex polygons and smoothing out the vertices. For each  $n$ , let  $P_n$  be the polygon with vertices  $\{v_1, \dots, v_n\}$ , where  $v_i$  is the unique point on  $\partial\Omega$  making an angle of  $2\pi i/n$  with the  $\xi_2$ -axis. Then  $P_n$  is convex and  $P_n \subset \Omega$ . Choose intervals  $I_n = [x_{n,0}, x_{n,1}] \subset \tilde{I}_n = (\tilde{x}_{n,0}, \tilde{x}_{n,1}) \subset \mathbb{R}$  centered at 0 such that  $\partial P_n \cap \{(\xi_1, \xi_2) : \xi_1 \in I_n, \xi_2 > 0\}$  can be parametrized as  $\{(\alpha, \tilde{\gamma}_n(\alpha)) : \alpha \in I_n\}$ , and also so that  $\{(\alpha, \tilde{\gamma}_n(\alpha)) : \alpha \in \tilde{I}_n\}$  does not contain any vertices of  $P_n$  except  $v_1$ .

Now let  $\eta \in C_0^\infty(\mathbb{R})$  be an even nonnegative function supported in  $(-1/2, 1/2)$  so that  $\int \eta(t) dt = 1$ . Let  $C_n = 100 \max\{(x_{n,0} - \tilde{x}_{n,0})^{-1}, (\tilde{x}_{n,1} - x_{n,1})^{-1}\}$ , and set

$$\gamma_n(\alpha) = \int C_n \eta(C_n t) \tilde{\gamma}_n(\alpha - t) dt, \quad \alpha \in I_n.$$

By the choice of  $C_n$ , we have that  $\{(\alpha, \gamma_n(\alpha)) : \alpha \in I_n\}$  coincides with  $P_n$  near the endpoints of  $I_n$ . We may thus obtain a smooth convex curve  $\partial\Omega_n$  by replacing  $\partial P_n$  near  $v_1$  with  $\{(\alpha, \gamma_n(\alpha)) : \alpha \in I_n\}$ , and then repeating the same procedure near the other vertices  $v_2, \dots, v_n$  after performing appropriate rotations.

It is clear that  $\{\Omega_n\}$  satisfies (1), (3), and (4), so it remains to show (2). Let  $\epsilon_0 > 0$  be sufficiently small so that for any  $\xi \in \partial\Omega$  and  $s_1, s_2 \in [1 - \epsilon_0, 1 + \epsilon_0]$ , the difference in slope between the tangent lines to the orbit  $\{t^A \xi : t > 0\}$  at  $s_1^A \xi$  and the tangent line at  $s_2^A \xi$  is less than  $\Theta(\Omega, A)/10$ . Now choose  $0 < \epsilon_1 < \epsilon_0$  sufficiently small so that

$$(2.5) \quad \{t^A \xi : t > 0, t \notin [1 - \epsilon_0, 1 + \epsilon_0], \xi \in \partial\Omega\} \\ \cap \{t^A \xi : t \in [1 - \epsilon_1, 1 + \epsilon_1], \xi \in \partial\Omega\} = \emptyset.$$

Next, choose  $N > 0$  sufficiently large so that whenever  $n \geq N$ , the following holds:

- (1)  $\partial\Omega_n \subset \{\xi : 1 - \epsilon_1 \leq \rho(\xi) \leq 1 + \epsilon_1\}$ ,
- (2) The difference in slope between the tangent line at any point  $x \in \partial\Omega_n$  and some supporting line of  $\partial\Omega$  at the vertex of  $P_n$  nearest to  $x$  is less than  $\Theta(\Omega, A)/10$ ,
- (3) For any  $\xi \in \partial\Omega_n$ , the difference in slope between the tangent vector to the orbit  $\{t^A \xi : t > 0\}$  at  $\xi$  and the tangent vector to the orbit

$\{t^A v_i\}$  at  $v_i$ , where  $v_i$  is the vertex of  $P_n$  nearest to  $\xi$ , is less than  $\Theta(\Omega, A)/10$ .

To see that we may choose  $N$  so that (1) and (3) are satisfied is fairly obvious, and to see that we may choose  $N$  so that (2) holds requires only a straightforward application of (2) from Lemma 2.1. It is easy to see that (2) and (3) imply that  $\Theta(\Omega_n, A) > \Theta(\Omega, A)/2$ . (1) and (2.5) imply that  $\{t^A \xi : t > 0, t \notin [1 - \epsilon_0, 1 + \epsilon_0], \xi \in \partial\Omega\}$  does not intersect  $\partial\Omega_n$ . Given  $\xi \in \partial\Omega$ , let  $t(\xi) > 0$  be the smallest value of  $t$  such that  $t^{-A}\xi \in \partial\Omega_n$ . Then  $t(\xi) \in [1 - \epsilon_0, 1 + \epsilon_0]$ . But by the choice of  $\epsilon_0$ , any tangent line to  $\{t^A \xi : t \in [1 - \epsilon_0, 1 + \epsilon_0]\}$  makes an angle of at least  $\Theta(\Omega, A)/4$  with the tangent line to  $\partial\Omega_n$  at  $t^{-A}\xi$ , and by convexity of  $\partial\Omega_n$  there can be no  $t > t(\xi)$  such that  $t^{-A}\xi \in \partial\Omega_n$ . Thus  $(\Omega_n, A)$  is a compatible pair for  $n \geq N$ .

□

**Lemma 2.3.** *Suppose that Proposition 1.3 holds in the special case when  $\partial\Omega$  is smooth, with a constant depending only on  $M, \epsilon, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)$ , and  $\Theta(\Omega, A)$ . Then Proposition 1.3 holds in the full stated generality.*

*Proof of Lemma 2.3.* Let  $\{\Omega_n\}$  be a sequence of convex domains approximating  $\Omega$  as in Lemma 2.2, and suppose the statement of Proposition 1.3 holds in the special case of convex domains with smooth boundaries, with a constant depending only on the quantities listed in Lemma 2.3. Fix a Schwartz function  $f \in \mathcal{S}(\mathbb{R}^2)$ . Then for every  $\epsilon > 0$  and every  $0 < \delta < C$ , for  $n$  sufficiently large we have

$$\left\| \left( \int_0^\infty |\psi_{n,t} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \leq C_{\epsilon, M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)} \delta^{1/2-\epsilon} \|f\|_4,$$

where

$$\psi_{n,t}(x) = \mathcal{F}(\phi(\frac{\rho_n(\cdot)}{t}))(x), \quad \phi(\xi) = \Phi(\frac{\xi - 1}{\delta}).$$

Since  $\phi(\frac{\rho_n(\cdot)}{t}) \rightarrow \phi(\frac{\rho(\cdot)}{t})$  uniformly as  $n \rightarrow \infty$ , we have that  $\psi_{n,t} * f(x) \rightarrow \psi_t * f(x)$  pointwise as  $n \rightarrow \infty$ . By Fatou's lemma applied twice,

$$\begin{aligned} \left\| \left( \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 &\leq \liminf_n \left\| \left( \int_0^\infty |\psi_{n,t} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \\ &\leq C_{\epsilon, M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)} \delta^{1/2-\epsilon} \|f\|_4, \end{aligned}$$

as desired. □

3. AN  $L^2$  MAXIMAL FUNCTION ESTIMATE

In [5], Córdoba proved  $L^2$  bounds for the Nikodym maximal function in  $\mathbb{R}^2$ . These bounds were an important ingredient in [2] to prove Proposition 1.3 for the special case of the classical (radial) Bochner-Riesz means. To prove Proposition 1.3 in the full stated generality, we need a nonisotropic version of Córdoba's result. To this end, we will closely follow [5] to prove the following proposition.

**Proposition 3.1.** *Let  $N, \lambda > 0$  be real numbers, and let  $\mathcal{C}$  be the collection of all rectangles in  $\mathbb{R}^2$  with dimensions  $\lambda$  and  $N\lambda$ . Let*

$$\mathcal{C}_k = \{(2^k)^A R : R \in \mathcal{C}, k \in \mathbb{Z}\}.$$

Define a maximal operator  $M_{\lambda, N}$  by

$$M_{\lambda, N} f(x) = \sup_{x \in R \in \bigcup_k \mathcal{C}_k} \frac{1}{|R|} \int_R |f(y)| dy.$$

Then there is a constant  $\beta(\operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) > 0$  such that for every Schwartz function  $f \in \mathcal{S}(\mathbb{R}^2)$ ,

$$\|M_{\lambda, N} f\|_2 \lesssim_{\operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)} \log(N)^{\beta(\operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))} \|f\|_2.$$

*Proof.* In what follows, for any rectangle  $R$  and any integer  $k$ , we will let  $((2^k)^A R)^* := (2^k)^A (R^*)$ . Here  $R^*$  denotes the double dilate of  $R$ , where the dilation is taken from the center of  $R$ . Similarly, if  $\mathcal{R}$  denotes any collection of nonisotropic dilates of rectangles, then  $\mathcal{R}^* := \{R^* : R \in \mathcal{R}\}$ .

For each  $k \in \mathbb{Z}$ , define a maximal operator  $M_{\lambda, N, k}$  on Schwartz functions  $f \in \mathcal{S}(\mathbb{R}^2)$  by

$$(3.1) \quad M_{\lambda, N, k} f(x) = \sup_{x \in R \in \mathcal{C}_k} \frac{1}{|R|} \int_R |f(y)| dy.$$

It follows from rescaling the corresponding result from [5] that for every  $f \in \mathcal{S}(\mathbb{R}^2)$ ,

$$(3.2) \quad \|M_{\lambda, N, k} f\|_2 \lesssim \log(3N)^{1/2} \|f\|_2.$$

Now we combine the estimates for the  $M_k$  to prove an  $L^2$  estimate for  $M$ . For each  $(i, k)$  where  $1 \leq i \leq N$  and  $k \in \mathbb{Z}$ , define a maximal operator  $T^{i, k}$  by

$$T^{i, k} f(x) = \sup_{(2^{-k})^A x \in R \in \mathcal{R}_i} \frac{1}{|(2^k)^A R|} \int_{(2^k)^A R} |f(y)| dy$$

where  $\mathcal{R}_i$  denotes the collection of all rectangles with direction  $\pi i N^{-1}$  and dimensions  $\lambda \times N\lambda$ . Fix a Schwartz function  $f \in \mathcal{S}(\mathbb{R}^2)$ , and apply a standard

covering lemma to obtain for each  $(i, k)$  a sequence of rectangles  $\{R_n^{i,k}\} \subset \mathcal{R}_i$  pairwise disjoint such that

$$E_\alpha^{i,k} = \{x : T^{i,k}f(x) > \alpha\} \subset \bigcup_n ((2^k)^A (R_n^{i,k})^*),$$

$$\frac{1}{|(2^k)^A R_n^{i,k}|} \int_{(2^k)^A R_n^{i,k}} |f(y)| dy > \alpha.$$

Then

$$E_\alpha = \{x : M_{\lambda,N}f(x) > 4\alpha\} \subset \bigcup_{i,k} E_\alpha^{i,k}.$$

Let

$$\mathcal{H} = \bigcup_{i,k,n} (2^k)^A R_n^{i,k}.$$

Let  $\mathcal{H}'$  be a subcollection of  $\mathcal{H}$  such that

- (1) There are no  $R, R' \in \mathcal{H}'$  such that  $R' \subset R^*$ .
- (2) If  $R \in \mathcal{H} \setminus \mathcal{H}'$ , then there is  $R' \in \mathcal{H}'$  such that  $R \subset (R')^*$ .

(To see that such a subcollection exists, we simply enumerate the rectangles in  $\mathcal{H}$  as  $R_1, R_2, \dots$ , and at step  $i$  we add  $R_i$  to  $\mathcal{H}'$  if  $R_i$  is not contained in  $R_j^*$  for any  $j < i$  such that  $R_j \in \mathcal{H}'$ , and in this case if  $R_j \subset R_i^*$  for any  $j < i$  such that  $R_j \in \mathcal{H}'$ , we remove  $R_j$  from  $\mathcal{H}'$ .) Then

$$(3.3) \quad E_\alpha \subset \bigcup_{R \in \mathcal{H}'} R^{**}.$$

Let  $\mathcal{H}_k = \mathcal{H}' \cap \mathcal{C}_k$ . Fix an integer  $a > 0$  such that  $B(0, 2) \subset (2^a)^A B(0, 1)$ , where  $B(0, r)$  denotes the (Euclidean) ball of radius  $r$  centered at the origin. Let  $n_0 = \max\{k : \mathcal{H}_k \neq \emptyset\}$ . For every  $j \geq 0$ , let

$$\Delta_j = \bigcup_{\substack{n_0 - (j+1)(\log N)^a \\ < k \leq n_0 - j(\log N)^a}} \mathcal{H}_k.$$

For each  $j$  let  $A_j = \bigcup_{R \in \Delta_j} R$ . Then the family of sets  $\{A_j\}$  is “almost disjoint”, i.e.  $A_{j_1} \cap A_{j_2} = \emptyset$  if  $|j_1 - j_2| > 2$ . To see this, suppose that  $R \in \Delta_{j_1}$  and  $R' \in \Delta_{j_2}$  with  $j_1 < j_2 - 2$  and  $R \cap R' \neq \emptyset$ . Choose  $k$  such that  $R \in \mathcal{C}_k$ . Then  $(2^{-k})^A R \subset B(x, N\lambda)$  for some  $x \in (2^{-k})^A R'$ . But  $((2^{-k})^A R')^* \supset B(x, N\lambda)$ , and so  $R \subset R'^*$ , a contradiction.

Now, by (3.3) we have

$$(3.4) \quad E_\alpha \subset \bigcup_j A_j^{**}.$$



Let  $f_j = f \cdot \chi_{A_j}$ . Define a maximal function  $S_j$  for  $g \in \mathcal{S}(\mathbb{R}^2)$  by

$$S_j g(x) = \sup_{\substack{x \in R \in \bigcup_{n_0+2-(j+1)(\log N)^a \leq k \leq n_0+2-j(\log N)^a} C_k}} \frac{1}{|R|} \int_R g(y) dy.$$

It follows from (3.2) that  $S_j$  is bounded on  $L^2(\mathbb{R}^2)$  with operator norm  $\lesssim (\log N)^{a+1/2}$ . Now if  $x \in A_j^{**}$ , then there is  $R \in \Delta_j$  such that  $x \in R^{**}$ . Then,

$$S_j f_j(x) \geq \frac{1}{|R^{**}|} \int_{R^{**}} |f_j(y)| dy \geq \frac{1}{16} \frac{1}{|R|} \int_R |f_j(y)| dy \geq \frac{1}{16} \alpha.$$

Thus  $A_j^{**} \subset \{x : S_j f_j(x) \geq \frac{1}{16} \alpha\}$ , and so

$$|A_j^{**}| \lesssim (\log N)^{2a+1} \frac{\|f_j\|_2^2}{\alpha^2}.$$

It follows that

$$(3.5) \quad |E_\alpha| \leq \sum_j |A_j^{**}| \lesssim (\log N)^{2a+1} \frac{1}{\alpha^2} \sum_j \|f_j\|_2^2 \lesssim (\log N)^{2a+1} \frac{\|f\|_2^2}{\alpha^2}.$$

To obtain a strong type  $L^2$  estimate for  $M_{\lambda,N}$  from (3.5), we will need to first prove a weak  $(1,1)$  estimate for  $M_{\lambda,N}$  and interpolate. By comparison with the Hardy-Littlewood maximal function and rescaling, we have for every  $k$ ,

$$(3.6) \quad |\{x : M_{\lambda,N,k}(f)(x) > \alpha\}| \lesssim N \frac{\|f\|_1}{\alpha}.$$

We now repeat the above argument, using (3.6) in place of (3.2) and obtain the weak  $(1,1)$  estimate

$$(3.7) \quad |\{x : M_{\lambda,N} f(x) > 4\alpha\}| \lesssim N \frac{\|f\|_1}{\alpha}.$$

The result now follows by interpolation of (3.7), (3.5) and the trivial  $L^\infty$  estimate for  $M_{\lambda,N}$ .  $\square$

#### 4. A DECOMPOSITION OF $\mathbb{R}^2$

In this section, we will introduce a decomposition of  $\mathbb{R}^2$  that plays a similar role as the decomposition of  $\mathbb{R}^2$  provided in [2]. The decomposition from [2] can be viewed more or less as a decomposition of the annulus  $|\xi - 1| \leq \delta$  into  $\delta$ -thickened caps that can be approximated by  $\delta^{1/2} \times \delta$  rectangles, and dilated at different scales to cover the plane in an almost-disjoint fashion. Here we employ a different decomposition of the set  $|\rho(\xi) - 1| \leq \delta$  into rectangles of width  $\delta$  and length essentially between  $\delta$  and 1, so that on each rectangle,  $\partial\Omega$  may be viewed as sufficiently “flat” at scale  $\delta$ . This decomposition was introduced by [19] to prove  $L^p$  bounds for Bochner-Riesz multipliers associated to convex domains. We then dilate these rectangles

nonisotropically at different scales to cover the plane in an almost-disjoint fashion.

**Decomposition of  $\partial\Omega$ .** Before we describe the decomposition of  $\mathbb{R}^2$ , we first need to introduce a decomposition of  $\partial\Omega$  from [19]. This decomposition allows us to write  $\partial\Omega$  as a disjoint union of pieces on which  $\partial\Omega$  is sufficiently “flat”. Here, the pieces in the decomposition will play the role that the  $\delta^{1/2}$ -caps play in the radial case.

We inductively define a finite sequence of increasing numbers

$$\mathfrak{A}(\delta) = \{a_0, \dots, a_Q\}$$

as follows. Let  $a_0 = -1$ , and suppose  $a_0, \dots, a_{l-1}$  are already defined. If

$$(4.1) \quad (t - a_{l-1})(\gamma'_L(t) - \gamma'_R(a_{l-1})) \leq \delta \text{ for all } t \in (a_{l-1}, 1]$$

and  $a_{l-1} \leq 1 - 2^{-M}\delta$ , then let  $a_l = 1$ . If (4.1) holds and  $a_{l-1} > 1 - 2^{-M}\delta$ , then let  $a_l = a_{l-1} + 2^{-M}\delta$ . If (4.1) does not hold, define

$$a_l = \inf\{t \in (a_{l-1}, 1] : (t - a_{l-1})(\gamma'_L(t) - \gamma'_R(a_{l-1})) > \delta\}.$$

Now note that (4.1) must occur after a finite number of steps, since we have  $|\gamma'_L|, |\gamma'_R| \leq 2^{M-1}$ , which implies that  $|t - s||\gamma'_L(t) - \gamma'_R(s)| < \delta$  if  $|t - s| < \delta 2^{-M}$ . Therefore this process must end at some finite stage  $l = Q$ , and so it gives a sequence  $a_0 < a_1 < \dots < a_Q$  so that for  $l = 0, \dots, Q-1$

$$(4.2) \quad (a_{l+1} - a_l)(\gamma'_L(a_{l+1}) - \gamma'_R(a_l)) \leq \delta,$$

and for  $0 \leq j < Q-1$ ,

$$(4.3) \quad (t - a_l)(\gamma'_L(t) - \gamma'_R(a_l)) > \delta \quad \text{if } t > a_{l+1}.$$

For a given  $\delta > 0$ , this gives a decomposition of

$$\partial\Omega \cap \{x : -1 \leq x_1 \leq 1, x_2 < 0\}$$

into pieces

$$\bigsqcup_{l=0,1,\dots,Q-1} \{x \in \partial\Omega : x_1 \in [a_l, a_{l+1}]\}.$$

Now let  $\{i_0, i_1, \dots, a_{Q'}\}$  be a refinement of  $\{a_0, a_1, \dots, a_Q\}$  corresponding to a partition of  $[-1, 1]$  into intervals  $\{I_j\}$  with  $I_j = [i_j, b_{j+1}]$  such that each interval  $[a_l, a_{l+1}]$  is a union of  $\lesssim \log(\delta^{-1})$  of the intervals  $I_j$ , and so that  $|I_j|/2 \leq |b_{j+1}| \leq 2|I_j|$ . We then have a decomposition

$$\partial\Omega \cap \{x : -1 \leq x_1 \leq 1, x_2 < 0\} = \bigsqcup_{j=0,1,\dots,Q'} \{x \in \partial\Omega : x_1 \in I_j\},$$

where  $Q' \lesssim \log(\delta^{-1})Q$ .

**Decomposition of  $\mathbb{R}^2$ .** With the previous decomposition of  $\partial\Omega$  in mind, we proceed to give a decomposition of  $\mathbb{R}^2$ . To begin, we define a *nonisotropic sector* to be a region bounded by the origin and two orbits  $\{t^A\xi : t > 0\}$  and  $\{t^A\xi' : t > 0\}$  for any  $\xi, \xi' \in \mathbb{R}^2 \setminus \{0\}$ . Observe there is an integer  $N_M > 0$  such that

- (1) We can write  $\mathbb{R}^2 = \bigcup_{i=0}^{N_M} \mathcal{S}_i$ , where each  $\mathcal{S}_i$  is a nonisotropic sector and the  $\mathcal{S}_i$  are essentially disjoint.
- (2) For every  $i$ , there is a rotation  $\mathcal{R}_i$  such that  $\mathcal{R}_i(\partial\Omega \cap \mathcal{S}_i) \subset \{x : -1/2 \leq x_1 \leq 1/2\}$ ,  $\mathcal{R}_i(\partial\Omega \cap \mathcal{S}_j) \cap \{x : -1/2 < x_1 < 1/2\} = \emptyset$  for  $i \neq j$ , and  $\mathcal{R}_0$  is the identity map.

For each  $i$ , define  $\tilde{\mathcal{S}}_i$  to be the nonisotropic sector bounded by the orbits  $\{t^A\xi_i : t > 0\}$  and  $\{t^A\xi'_i : t > 0\}$  where  $\xi = (\xi_1, \xi_2)$  is the unique point in  $\mathcal{R}_i\partial\Omega$  with  $\xi_1 = -1$  and  $\xi_2 > 0$ , and  $\xi' = (\xi'_1, \xi'_2)$  is the unique point in  $\mathcal{R}_i\partial\Omega$  with  $\xi_1 = 1$  and  $\xi_2 > 0$ . Clearly,  $\mathcal{S}_i \subset \tilde{\mathcal{S}}_i$ . Let  $\{(\alpha, \gamma_i(\alpha)) : \alpha \in [-1, 1]\}$  be a parametrization of  $\mathcal{R}_i(\partial\Omega \cap \tilde{\mathcal{S}}_i)$ . For  $0 \leq i \leq N_M$ , let  $R_i$  denote the region of  $\mathbb{R}^2$  bounded by the level sets  $\{x : \rho(x) = 1/2\}$  and  $\{x : \rho(x) = 2\}$  and the nonisotropic sector  $\tilde{\mathcal{S}}_i$ . Similarly, let  $R'_i$  denote the region of  $\mathbb{R}^2$  bounded by the level sets  $\{x : \rho(x) = 1/4\}$  and  $\{x : \rho(x) = 4\}$  and the nonisotropic sector  $\tilde{\mathcal{S}}_i$ . Fix  $\delta > 0$ . Let  $R_{i,\delta}$  denote the region bounded by the level sets  $\{x : \rho(x) = 1 - 2\delta\}$  and  $\{x : \rho(x) = 1 + 2\delta\}$  and  $\tilde{\mathcal{S}}_i$ . Note that  $\bigcup_{i=0}^{N_M} R_{i,\delta}$  contains the support of  $\mathcal{F}[\psi_1]$ , where  $\psi_1$  is as in Proposition 1.3.

Recall the previous decomposition of  $[-1, 1]$  into intervals  $\{I_j\}$ . Let  $B_{i,j,0,0}$  denote the region bounded by  $R_{i,\delta}$  and the orbits  $\{t^A\mathcal{R}_i^{-1}(i_j, \gamma_i(i_j)) : t > 0\}$  and  $\{t^A\mathcal{R}_i^{-1}(b_{j+1}, \gamma_i(b_{j+1})) : t > 0\}$ , so that  $R_{i,\delta} = \bigcup_j B_{i,j,0,0}$ . For each integer  $m$ , let  $B_{i,j,m,0} = (\frac{1+2\delta}{1-2\delta})^{mA} B_{i,j,0,0}$ . Now let  $N_\delta, N'_\delta$  be integers such that

$$R_i \subset \bigcup_{j, N_\delta \leq m \leq N'_\delta} B_{i,j,m,0} \subset R'_i.$$

We are now ready to state our decomposition of  $\mathbb{R}^2$ . For each integer  $n$ , let  $B_{i,j,m,n} = (2^{2n})^A(B_{i,j,m,0})$ . Then

$$\mathcal{S}_i \subset \bigcup_{j, N_\delta \leq m \leq N'_\delta, n \in \mathbb{Z}} B_{i,j,m,n},$$

$$\mathbb{R}^2 = \bigcup_{0 \leq i \leq N_M, j, N_\delta \leq m \leq N'_\delta, n \in \mathbb{Z}} B_{i,j,m,n},$$

and there is an integer  $N'_M$  depending only on  $M$  such that every point of  $\mathbb{R}^2$  lies in at most  $N'_M$  many elements of the collection  $\{B_{i,j,m,n}\}$ .

**Some important properties of the decomposition.** We now prove some essential geometric facts regarding our decomposition; these may be viewed as analogs of Proposition 3 parts (i) – (iii) from [2]. The following proposition is a key fact regarding the almost disjointness of algebraic sums of the pieces in our decomposition.

**Proposition 4.1.** *For a constant  $C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) > 0$  depending only on  $M$  and the eigenvalues of  $A$ , let*

$$\mathcal{T}_0 = \{\xi : 1/4 \leq \rho(\xi) \leq 4, |\xi_1| \leq C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\},$$

$$\mathcal{T}_1 = \bigcup_{k \in \mathbb{Z}} (2^{4k})^A(\mathcal{T}_0).$$

For  $0 < t < \infty$ , let

$$\mathcal{A}_t = \{B \in \{B_{i,j,m,n}\} : \exists \xi \in B \text{ with } \rho(\xi) = 1/t \text{ and } \xi \in \mathcal{T}_1\}.$$

Fix positive real numbers  $u$  and  $t$  satisfying  $1/2 < u/t < 2$  with  $u \in \bigcup_{k \in \mathbb{Z}} [2^{4k-1}, 2^{4k+1}]$ , and let  $\mathcal{B}_{u,t}$  denote the collection of all sets of the form  $\{A + B\}_{A \in \mathcal{A}_t, B \in \mathcal{A}_u}$ . Then if  $C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  is chosen sufficiently small, there exists a constant  $C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) > 0$  (depending only on  $M$  and the eigenvalues of  $A$  and independent of  $\delta$  and the choice of  $u$  and  $t$ ) such that every point of  $\mathbb{R}^2$  is contained in at most

$$C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))(\log(\delta^{-1}))^2$$

elements of  $\mathcal{B}_{u,t}$ .

*Proof.* Without loss of generality, assume that  $u = 1$ . For any  $v \in A \in \mathcal{A}_t$  and  $w \in B \in \mathcal{A}_u$ , let  $\sigma^+(v, w)$  denote the minimum nonnegative difference in slope between supporting lines to the convex curve

$$\Sigma_v := \{\xi : \rho(\xi) = \rho(v), \xi \in \mathcal{T}_1\}$$

at  $v$  and supporting lines to the convex curve

$$\Sigma_w := \{\xi : \rho(\xi) = \rho(w), \xi \in \mathcal{T}_1\}$$

at  $w$ , and  $\sigma^+(v, w) := +\infty$  if no nonnegative difference exists. Let  $\sigma^-(v, w)$  denote the maximum nonpositive difference in slope between supporting lines to  $\Sigma_v$  at  $v$  and supporting lines to  $\Sigma_w$  at  $w$ , and  $\sigma^-(v, w) := -\infty$  if no nonpositive difference exists. Note that for every  $(v, w)$  at least one of  $\sigma^+(v, w)$  and  $\sigma^-(v, w)$  is finite, and if  $C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  is sufficiently small, then the slope of any supporting line is between  $-2^{2M}$  and  $2^{2M}$ . Given  $x \in \mathcal{B}_{u,t}$ , we have one of three cases:

- (1) There is  $v \in A \in \mathcal{A}_t$  and  $w \in B \in \mathcal{A}_u$  with  $v + w = x$  and  $\sigma^+(v, w)$  finite, but  $\sigma^-(v, w)$  is infinite for every pair  $(v', w')$  with  $v \in A' \in \mathcal{A}_t$ ,  $w \in B' \in \mathcal{A}_u$ , and  $v' + w' = x$ ,

- (2) There is  $v \in A \in \mathcal{A}_t$  and  $w \in B \in \mathcal{A}_u$  with  $v + w = x$  and  $\sigma^-(v, w)$  finite, but  $\sigma^+(v, w)$  is infinite for every pair  $(v', w')$  with  $v \in A' \in \mathcal{A}_t$ ,  $w \in B' \in \mathcal{A}_u$ , and  $v' + w' = x$ ,
- (3) There is  $v \in A \in \mathcal{A}_t$  and  $w \in B \in \mathcal{A}_u$  with  $v + w = x$  and  $\sigma^+(v, w)$  finite, and there is  $v' \in A' \in \mathcal{A}_t$  and  $w' \in B' \in \mathcal{A}_u$  with  $v' + w' = x$  and  $\sigma^-(v', w')$  finite.

Let us assume we have case 1. Given  $x \in \mathbb{R}^2$ , choose  $v = (v_1, v_2) \in A \in \mathcal{A}_t$  and  $w = (w_1, w_2) \in B \in \mathcal{A}_u$  with  $v + w = x$  minimizing  $\sigma^+(v, w)$ . Now suppose there is  $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in \tilde{A} \in \mathcal{A}_t$  and  $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \tilde{B} \in \mathcal{B}_t$  such that  $\tilde{v} + \tilde{w} = x$ . Since  $\Sigma_v$  and  $\Sigma_w$  are convex, we have

$$\tilde{v}_1 \leq v_1 + C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta,$$

$$\tilde{w}_1 \geq w_1 - C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta,$$

where  $C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) > 0$  is a constant that depends only on  $M$  and the eigenvalues of  $A$ . Thus

$$(4.4) \quad v_1 - \tilde{v}_1 = \tilde{w}_1 - w_1 \geq -C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta.$$

Choose indices  $i_0, j_0, m_0, n_0$  and  $i'_0, j'_0, m'_0, n'_0$  such that  $B_{i_0, j_0, m_0, n_0} \ni v$  and  $B_{i'_0, j'_0, m'_0, n'_0} \ni w$ . (There are  $\lesssim_{M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)}$  possible choices of indices.) Also choose indices  $i_1, j_1, m_1$  and  $i'_1, j'_1, m'_1$  such that  $B_{i_1, j_1, m_1, n_1} \ni \tilde{v}$  and  $B_{i'_1, j'_1, m'_1, n'_1} \ni \tilde{w}$ . Note that we must necessarily have  $m_1 = m_0$  and  $m'_1 = m'_0$ , and also that  $-10 \leq n_1, n'_1 \leq 10$ . We next observe that for some sufficiently large constant  $C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  we must have

$$\begin{aligned} j_0 - C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) \log(\delta^{-1})^2 &\leq j_1 \\ &\leq j_0 + C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) \log(\delta^{-1})^2, \end{aligned}$$

$$\begin{aligned} j'_0 - C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) \log(\delta^{-1})^2 &\leq j'_1 \\ &\leq j'_0 + C'''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) \log(\delta^{-1})^2, \end{aligned}$$

since otherwise (4.3) and (4.4) would imply that  $\tilde{v}_2 + \tilde{w}_2 < v_2 + w_2 - C''''(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta$ . This completes the proof for case 1, since we have shown that for some constant  $C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  sufficiently large there are fewer than  $C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) \log(\delta^{-1})^2$  possible choices of indices  $i_1, j_1, m_1, n_1$  and  $i'_1, j'_1, m'_1, n'_1$  such that  $B_{i_1, j_1, m_1, n_1} \ni \tilde{v}$  and  $B_{i'_1, j'_1, m'_1, n'_1} \ni \tilde{w}$ . The proof for case 2 is similar.

Now let us assume we have case 3. Suppose there is  $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in \tilde{A} \in \mathcal{A}_t$  and  $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \tilde{B} \in \mathcal{B}_t$  such that  $\tilde{v} + \tilde{w} = x$ . Then if  $\sigma^+(\tilde{v}, \tilde{w})$  is finite, then there is a constant  $C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) > 0$  such that

$$\tilde{v}_1 \leq v_1 + C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta, \quad \tilde{w}_1 \geq w_1 - C'(b_j M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta,$$

and if  $\sigma^-(\tilde{v}, \tilde{w})$  is finite, then

$$\tilde{v}_1 \leq v_1 + C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta, \quad \tilde{w}_1 \geq w_1 - C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))\delta.$$

In either case, the previous argument shows there is a constant  $C = C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) > 0$  such that there are fewer than  $C \log(\delta^{-1})^2$  possible choices of indices  $i_1, j_1, m_1, n_1$  and  $i'_1, j'_1, m'_1, n'_1$  such that  $B_{i_1, j_1, m_1, n_1} \ni \tilde{v}$  and  $B_{i'_1, j'_1, m'_1, n'_1} \ni \tilde{w}$ .

□

**Proposition 4.2.** *Let  $N(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  be a positive integer and let  $\delta > 0$ , and fix positive real numbers  $u$  and  $t$  satisfying  $\delta^{N(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))} t > u$ . Then if  $N(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  is sufficiently large, there exists a constant  $C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)) > 0$  (independent of  $\delta$  and the choice of  $u$  and  $t$ ) such that no point of  $\mathbb{R}^2$  is contained in more than  $C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  of the sets  $\{A + B_\rho(0, 2/t)\}_{A \in \mathcal{A}_u}$ , where  $B_\rho(0, r) = \{x \in \mathbb{R}^2 : \rho(x) \leq r\}$ .*

*Proof.* Without loss of generality, suppose that  $u = 1$ . Fix  $A \in \mathcal{A}_u$ , and let  $x \in A$  and let  $y \in B_\rho(0, 2/t)$ . Choose  $N(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$  large enough to make  $B_\rho(0, 2/t) \subset B(0, \delta^2)$ , where  $B(0, \delta^2)$  denotes the (Euclidean) ball of radius  $\delta^2$  centered at the origin. Assume  $\delta < C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A))$ , where  $C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)) > 0$  is chosen sufficiently small so that the minimum angle between the tangent line to  $\xi \in \partial\Omega$  and any tangent line to the curve  $\{t^A \xi : 1 - 10\delta \leq t \leq 1 + 10\delta\}$  is at least  $\delta^{1/2}$ . Now for any  $\xi \in \partial\Omega$ ,  $1 - 10\delta \leq t \leq 1 + 10\delta$ , we have

$$\left| \frac{d}{dt}(t^A \xi) \right| = |t^{-1} A t^A \xi| \gtrsim_{M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2)} 1,$$

and it follows that if  $C'(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A))$  is sufficiently small, the (Euclidean) distance between  $t^A \xi$  and the tangent line to  $\partial\Omega$  at  $\xi$  is at least  $10\delta^2$ . Since  $\Omega$  is convex, we conclude that the distance between  $\partial\Omega$  and  $(1 + \delta)^A \partial\Omega$  is at least  $10\delta^2$ . Similarly, the distance between  $\partial\Omega$  and  $(1 - \delta)^A \partial\Omega$  is at least  $10\delta^2$ . It follows there is an absolute constant  $C$  such that for any given  $\xi \in \mathbb{R}^2$ , there are fewer than  $C$  possible values of  $m$  (and clearly also fewer than  $C$  possible values of  $n$ ) such that  $B_{i, j, m, n} + B(0, \delta^2) \ni \xi$  for some  $B_{i, j, m, n} \in \mathcal{A}_1$ . It remains to obtain an upper bound for the number of possible values of  $j$ . But it is clear that  $\operatorname{dist}(B_{i, j, m, n}, B_{i, j', m, n}) \geq \delta/10$  for  $|j - j'| > 2$ , and this finishes the proof.

□

**Proposition 4.3.** *There exists an absolute constant  $C > 0$  such that for each fixed quadruple  $(i, j, m, n)$ , the logarithmic measure of  $\{t : B_{i, j, m, n} \cap \operatorname{supp} \mathcal{F}[\psi_t] \neq \emptyset\}$  is less than or equal to  $C\delta$ .*

*Proof.* Immediate.

□

5. KERNEL ESTIMATES AND ANOTHER  $L^2$  MAXIMAL FUNCTION ESTIMATE

We note that in both [4] and [2], it was important that regarding the decomposition of the multiplier  $\phi(\delta^{-1}(1 - |\xi|))$  where  $\phi$  was a smooth bump function into pieces supported on  $\delta^{1/2} \times \delta$  rectangles, each piece of the multiplier had  $L^1$  norm essentially 1. This was also true of the decomposition of  $|\rho(\xi) - 1| \leq \delta$  introduced in [19]. In this section we prove that after the introduction of nonisotropic dilations, the same holds true.

The argument presented in [2] also used  $L^2$  bounds for maximal functions given by the supremum of convolutions by smooth bumps supported on finitely many essentially disjoint pieces of the decomposition of  $\mathbb{R}^2$  given in [2]. Since these smooth bumps could be dominated by Schwartz functions adapted to rectangles, such a maximal function could be dominated by a Nikodym maximal function. Here, as well as in [19], we do not have domination of the functions in our partition of unity by Schwartz functions adapted to rectangles, and the proof of  $L^1$  kernel estimates is more delicate. As in [19], this also implies that the associated maximal function that we use is not simply a nonisotropic Nikodym maximal function. However, we will show that the  $L^2$  bounds for the nonisotropic Nikodym maximal function proved earlier imply  $L^2$  bounds for the maximal function that we are interested in, with a similar constant.

**A partition of unity associated to the decomposition of  $\mathbb{R}^2$ .** First, we need to define a partition of unity of  $\mathbb{R}^2$ , and as mentioned above one goal of this section is to show that each function in our partition of unity has bounded  $L^1$  norm. Recall the decomposition

$$\mathbb{R}^2 = \bigcup_{i,j,m,n} B_{i,j,m,n}.$$

We now introduce a partition of unity  $\{\sigma_{i,j,m,n}\}$  such that

- (1)  $\sigma_{i,j,m,n} \in C^\infty(\mathbb{R}^2)$  for every  $(i, j, m, n)$ ,
- (2)  $\sum_{i,j,m,n} \sigma_{i,j,m,n}(x) = 1$  for every  $x \in \mathbb{R}^2$ ,
- (3) There is a constant  $C_M$  such that for every  $(i_0, j_0, m_0, n_0)$ ,  $\sigma_{i_0,j_0,m_0,n_0}$  is supported in  $\bigcup_{|j|,|m| \leq C_M} B_{i_0,j_0+j,m_0+m,n_0}$ .

Let  $\phi \in C^\infty([-1, 1])$  be nonnegative and identically 1 on  $[-1/2, 1/2]$ , and for  $n \in \mathbb{Z}$  set  $\phi_n(\cdot) = \phi(2^{-n-1}\cdot) - \phi(2^{-n}\cdot)$ . For each  $m$ , let  $\psi_m \in C^\infty(1 - (2m+10)\delta, 1 + (2m+10)\delta)$  such that  $\sum_m \psi_m$  is identically 1 on the support of  $\phi_0$ , and for every  $k$ ,  $D^k \psi_m \lesssim_k \delta^{-k}$ .

For each  $i$ , let  $S_i$  be the *isotropic* sector bounded by  $|\xi| = 2$ ,  $|\xi| = 2^{M+2}$ , and the rays through the origin and the points  $\xi$  and  $\xi'$ , where  $\xi = (\xi_1, \xi_2)$  is the unique point in  $\mathcal{R}_i \partial \Omega$  with  $\xi_1 = -1/4$  and  $\xi_2 > 0$ , and  $\xi' = (\xi'_1, \xi'_2)$  is

the unique point in  $\mathcal{R}_i\partial\Omega$  with  $\xi_1 = 1/4$  and  $\xi_2 > 0$ . Let  $\tilde{S}_i$  be the isotropic sector bounded by  $|\xi| = 1$ ,  $|\xi| = 2^{M+3}$ , and the rays through the origin and the points  $\xi$  and  $\xi'$ , where  $\xi = (\xi_1, \xi_2)$  is the unique point in  $\mathcal{R}_i\partial\Omega$  with  $\xi_1 = -3/4$  and  $\xi_2 > 0$ , and  $\xi' = (\xi'_1, \xi'_2)$  is the unique point in  $\mathcal{R}_i\partial\Omega$  with  $\xi_1 = 3/4$  and  $\xi_2 > 0$ . For each  $i$ , let  $\Psi_i$  be a smooth function supported in  $\tilde{S}_i$  and identically 1 on  $S_i$ , such that  $D^k\Psi_i \lesssim_{M,k} 1$  for all  $k$  and  $\sum_i \Psi_i$  is identically 1 on the region bounded by  $|\xi| = 2$  and  $|\xi| = 2^{M+2}$ .

Fix  $i$ , and for each  $j$ , let  $\ell_{j-1}$ ,  $\ell_j$ , and  $\ell_{j+1}$  be the lines through  $(b_{j-1}, \gamma_i(b_{j-1}))$ ,  $(i_j, \gamma_i(i_j))$ , and  $(b_{j+1}, \gamma_i(b_{j+1}))$ , respectively, with slopes orthogonal to the tangent vectors  $(1, \gamma'_i(b_{j-1}))$ ,  $(1, \gamma'_i(i_j))$ , and  $(1, \gamma'_i(b_{j+1}))$ , respectively. Let  $e_j$  be a unit vector orthogonal to  $\ell_j$ . Let  $\alpha$  be a  $C^\infty(\mathbb{R})$  function such that  $0 \leq \alpha \leq 1$ ,  $\alpha(x) = 1$  for  $x \in [-1, 1]$  and  $\alpha(x) = 0$  for  $x \notin [-\frac{101}{100}, \frac{101}{100}]$ , and set  $\alpha_j(\xi) = \alpha(|I_j|^{-1}(\xi - (i_j, \gamma_i(i_j))) \cdot e_j)$ . We are now ready to define the functions  $\sigma_{i,j,m,n}$ . Let

$$(5.1) \quad \sigma_{i,j,m,0}(\xi) = \phi_0(\rho(\xi))\Psi_i\left(\left(\frac{1-2\delta}{1+2\delta}\right)^{mA}\xi\right)\psi_m(\rho(\xi)) \\ \times \alpha_j\left(\mathcal{R}_i\left(\frac{1-2\delta}{1+2\delta}\right)^{mA}\xi\right)(1 - \alpha_{j+1}(\mathcal{R}_i\left(\frac{1-2\delta}{1+2\delta}\right)^{mA}\xi)),$$

and

$$(5.2) \quad \sigma_{i,j,m,n}(\xi) = \sigma_{i,j,m,0}((2^{-n})^A\xi).$$

For every  $i$  and every  $m$ , we have

$$\sum_j \alpha_j(\mathcal{R}_i\left(\frac{1-2\delta}{1+2\delta}\right)^{mA}\xi)(1 - \alpha_{j+1}(\mathcal{R}_i\left(\frac{1-2\delta}{1+2\delta}\right)^{mA}\xi))$$

is identically 1 on the support of

$$\phi_0(\rho(\xi))\Psi_i\left(\left(\frac{1-2\delta}{1+2\delta}\right)^{mA}\xi\right)\psi_m(\rho(\xi)),$$

and since

$$\sum_i \sum_m \phi_0(\rho(\xi))\Psi_i\left(\left(\frac{1-2\delta}{1+2\delta}\right)^{mA}\xi\right)\psi_m(\rho(\xi)) \\ = \sum_m \phi_0(\rho(\xi))\psi_m(\rho(\xi)) = \phi_0(\rho(\xi)),$$

it follows that for every  $\xi \in \mathbb{R}^2$ ,

$$\sum_{i,j,m,n} \sigma_{i,j,m,n}(\xi) = 1.$$

**Introduction of a maximal function associated with the partition of unity.** Let

$$K_{i,j,m,n}(x) = \mathcal{F}[\sigma_{i,j,m,n}(\cdot)](x).$$



We define a maximal function  $\overline{M}$  on  $f \in \mathcal{S}(\mathbb{R}^2)$  by

$$\overline{M}f(x) = \sup_{i,j,m,n} \sup_{2^{n-10} \leq t \leq 2^{n+10}} |\psi_t * K_{i,j,m,n} * f(x)|.$$

We will prove the following  $L^2$  bounds for  $\overline{M}$ .

**Proposition 5.1.** *Let  $\epsilon > 0$ . There is a constant  $C = C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A))$  such that if  $0 < \delta < C$ , then for  $f \in \mathcal{S}(\mathbb{R}^2)$ ,*

$$\|\overline{M}f\|_{L^2(\mathbb{R}^2)} \lesssim_{\epsilon, M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)} \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}.$$

*Proof.* The proof will follow [19]. First note that without loss of generality we may drop the “sup” in the  $i$  index in the definition of  $\overline{M}$  and assume  $i = 0$ , and so in what follows we drop all  $i$ -indices. Set  $l = \lceil \log(\delta^{-1}) \rceil$ . We decompose  $\overline{M} = M_1 + M_2$ , where

$$M_1 f(x) = \sup_{j,m,n} \sup_{2^{n-10} \leq t \leq 2^{n+10}} |\psi_t * (K_{j,m,n} \cdot \chi_{|t^A| \geq 2^{10M \cdot l}}) * f(x)|,$$

$$M_2 f(x) = \sup_{j,m,n} \sup_{2^{n-10} \leq t \leq 2^{n+10}} |\psi_t * (K_{j,m,n} \cdot \chi_{|t^A| < 2^{10M \cdot l}}) * f(x)|.$$

We will first prove Proposition 5.1 with  $\overline{M}$  replaced by  $M_1$ . Let  $\sigma_j(\xi) = \mathcal{F}^{-1}[K_{j,0,0}(\cdot)](\xi)$ . Note that

$$(5.3) \quad \sigma_j(\xi) = \phi_0(\rho(\xi))\psi_0(\rho(\xi))m_j(\xi),$$

where

$$(5.4) \quad m_j(\xi) = \Psi_0(\xi)\phi_0(2^{-2M}\xi)\alpha_j(\xi)(1 - \alpha_j(\xi - 2^{M+10}|I_j|(1, \gamma'_0(i_j)))) \\ (1 - \alpha_{j+1}(\xi))(\alpha_{j+1}(\xi + 2^{M+10}|I_j|(1, \gamma'_0(b_{j+1}))))).$$

Now let  $\beta \in C^\infty$  be supported in  $[-1, 1]$ , and let  $h_l(s) = \beta(2^l(1 - s))$ . Note that (5.3) says that  $\sigma_j$  is of the form  $h_l(\rho(\cdot))m_j(\cdot)$ . We claim that to prove Proposition 5.1 with  $M_1$  in place of  $\overline{M}$ , it in fact suffices to prove Proposition 5.1 with  $\overline{M}f$  replaced by

$$\sup_{t \in (0, \infty)} |(\chi_{|t^A| \geq 2^{5M \cdot l}} \cdot \mathcal{F}^{-1}[h_l(t\rho(\cdot))]) * f(x)|.$$

This will follow immediately from the observation that  $2^{M+10}|I_j|^{-1} \ll 2^{10M \cdot l}$  and that for any annulus  $\mathcal{A}_k$ ,

$$(5.5) \quad \int_{\mathcal{A}_k} \mathcal{F}^{-1}[h_l(\rho(\cdot))](x) dx \lesssim 1,$$

which will be proven later.

We now prove pointwise estimates for  $\mathcal{F}^{-1}[h_l(\rho(\cdot))](x)$ , which we write as an integral over  $\partial\Omega$  as follows:

$$(2\pi)^2 \mathcal{F}^{-1}[h_l(\rho(\cdot))](x) = \int_{\Omega} h_l(\rho(\xi)) e^{i\langle x, \xi \rangle} d\xi = - \int_{\Omega} e^{i\langle x, \xi \rangle} \int_{\rho(\xi)}^{\infty} h'_l(s) ds d\xi$$

$$\begin{aligned}
&= - \int_0^\infty h'_l(s) \int_{\rho(\xi) \leq s} e^{i\langle x, \xi \rangle} d\xi ds = - \int_0^\infty h'_l(s) \int_{\rho(\xi) \leq 1} e^{i\langle x, s^A \xi \rangle} |\det s^A| d\xi dx \\
&= \int_0^\infty i |s^{A^*} x|^{-2} h'_l(s) \int_{\partial\Omega} e^{i\langle s^{A^*} x, \xi \rangle} \langle s^{A^*} x, n(\xi) \rangle d\sigma(\xi) |\det s^A| ds.
\end{aligned}$$

In the above computation, we used the divergence theorem applied to the vector field  $\xi \mapsto (i |s^{A^*} x|^2)^{-1} s^{A^*} x e^{i\langle s^{A^*} x, \xi \rangle}$ . For each  $i$ , let  $\zeta_i \in C^\infty(\mathbb{R})$  be supported in  $[-4/5, 4/5]$  and identically 1 on  $[-1/3, 1/3]$  such that  $\sum_i \zeta_i((\mathcal{R}_i(1/\rho(\xi))^A \xi)_1) \equiv 1$ . It suffices to estimate

$$\begin{aligned}
(5.6) \quad & \int_0^\infty i |s^{A^*} x|^{-2} h'_l(s) \int_{\partial\Omega} e^{i\langle s^{A^*} x, \xi \rangle} \\
& \langle s^{A^*} x, n(\xi) \rangle \zeta_0((\xi)_1) d\sigma(\xi) |\det s^A| ds.
\end{aligned}$$

We introduce homogeneous coordinates

$$(5.7) \quad (s, \alpha) \mapsto \xi(s, \alpha) = s^A(\alpha, \gamma_0(\alpha)).$$

The Jacobian of the map (5.7) is

$$\langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle.$$

Using homogeneous coordinates, (5.6) can be written as

$$\begin{aligned}
(5.8) \quad & i \int \zeta_0(\alpha) \int_0^\infty |s^{A^*} x|^{-2} h'_l(s) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} \\
& \times \langle x, s^A(-\gamma'_0(\alpha), 1)(1 + (\gamma'_0(\alpha))^2)^{-1/2} \rangle \\
& \times \langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle |\det s^A| ds d\alpha.
\end{aligned}$$

Let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function supported in  $[-\epsilon, \epsilon]$ , where

$$(5.9) \quad \epsilon = \Theta(\Omega, A) \cdot \min(|\lambda_1|, |\lambda_2|) / (100 \cdot 2^{M+2}).$$

Then (5.8) can be written as  $\tilde{K}_1(x) + \tilde{K}_2(x)$ , where

$$\begin{aligned}
\tilde{K}_1(x) &= i \int \zeta_0(\alpha) \int_0^\infty |s^{A^*} x|^{-2} h'_l(s) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} \\
& \times \eta\left(\frac{\langle x, A(\alpha, \gamma_0(\alpha)) \rangle}{|x|}\right) \langle x, s^A(-\gamma'_0(\alpha), 1)(1 + (\gamma'_0(\alpha))^2)^{-1/2} \rangle \\
& \times \langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle |\det s^A| ds d\alpha,
\end{aligned}$$

$$\begin{aligned}
\tilde{K}_2(x) &= i \int \zeta_0(\alpha) \int_0^\infty |s^{A^*} x|^{-2} h'_l(s) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} \\
& \times (1 - \eta\left(\frac{\langle x, A(\alpha, \gamma_0(\alpha)) \rangle}{|x|}\right)) \langle x, s^A(-\gamma'_0(\alpha), 1)(1 + (\gamma'_0(\alpha))^2)^{-1/2} \rangle \\
& \times \langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle |\det s^A| ds d\alpha.
\end{aligned}$$

To estimate  $\tilde{K}_2(x)$ , we integrate by parts with respect to  $s$  twice. This yields

$$\tilde{K}_2(x) = i \int \zeta_0(\alpha) \left( 1 - \eta\left(\frac{\langle x, A(\alpha, \gamma_0(\alpha)) \rangle}{|x|}\right) \right) \int_0^\infty g_2(x, s, \alpha) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} ds d\alpha,$$

where

$$\begin{aligned} g_2(x, s, \alpha) = & \frac{d}{ds} \left( \langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle^{-1} \frac{d}{ds} \left( \langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle^{-1} \right. \right. \\ & \times |s^{A^*} x|^{-2} h'_l(s) \left\langle x, s^A(-\gamma'_0(\alpha), 1)(1 + (\gamma'_0(\alpha))^2)^{-1/2} \right\rangle \\ & \left. \left. \times \langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle |\det s^A| \right) \right). \end{aligned}$$

Note that if  $0 < \delta < C$  for a sufficiently small constant  $C > 0$ , then for  $s$  in the support of  $h_l(s)$  and for  $x$  in the support of  $1 - \eta\left(\frac{\langle x, A(\alpha, \gamma_0(\alpha)) \rangle}{|x|}\right)$ , we have  $\langle x, s^{-1} s^A A(\alpha, \gamma_0(\alpha)) \rangle \geq |x| \cdot \epsilon/2$ . Thus

$$|g_2(x, s, \alpha)| \lesssim |x|^{-3} |h''_l(s)|.$$

This implies that

$$(5.10) \quad |\tilde{K}_2(x)| \lesssim |x|^{-3} \int \zeta_0(\alpha) \int |h''_l(s)| ds d\alpha \lesssim 2^l |x|^{-3}.$$

To estimate  $\tilde{K}_1(x)$ , we integrate by parts with respect to  $\alpha$  once and then with respect to  $s$  twice, which yields

$$\tilde{K}_1(x) = \int_0^\infty \int g_1(x, s, \alpha) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} d\alpha ds,$$

where

$$\begin{aligned} g_1(x, s, \alpha) = & -\frac{d}{ds} \left( \langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle^{-1} \frac{d}{ds} \left( \langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle^{-1} \right. \right. \\ & \times \frac{d}{d\alpha} \left( \langle x, s^A(1, \gamma'_0(\alpha)) \rangle^{-1} \zeta_0(\alpha) \eta\left(\frac{\langle x, A(\alpha, \gamma_0(\alpha)) \rangle}{|x|}\right) \right. \\ & \times i \left\langle x, s^A(-\gamma'_0(\alpha), 1)(1 + (\gamma'_0(\alpha))^2)^{-1/2} \right\rangle \\ & \left. \left. \times \langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle |s^{A^*} x|^{-2} h'_l(s) |\det s^A| \right) \right) \right). \end{aligned}$$

By the choice of  $\epsilon$ , we have that for  $s$  in the support of  $h_l(s)$  and for  $x$  in the support of  $\eta\left(\frac{\langle x, A(\alpha, \gamma_0(\alpha)) \rangle}{|x|}\right)$ , if  $\theta$  denotes the angle between  $x$  and  $A(\alpha, \gamma_0(\alpha))$ , then  $\cos(\theta) \leq \Omega(\Theta, A)/100$ . Since  $A(\alpha, \gamma_0(\alpha))$  is tangent to the

orbit  $\{s^A(\alpha, \gamma_0(\alpha)) : s > 0\}$  at  $(\alpha, \gamma_0(\alpha))$ , if  $0 < \delta < C$  for a sufficiently small constant  $C$ , we have

$$|\langle x, s^A(1, \gamma'_0(\alpha)) \rangle|^{-1} \geq (\Theta(\Omega, A)/2^{M+2}) \cdot |x|.$$

It follows that

$$|g_1(x, s, \alpha)| \lesssim |x|^{-2} |\langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle|^{-2} |h_l''(s)| \zeta_0(\alpha) (1 + |\gamma''(\alpha)|),$$

and hence

$$(5.11) \quad |\tilde{K}_1(x)| \lesssim \int \int |x|^{-2} |1 + |\langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle||^{-2} \times |h_l''(s)| \zeta_0(\alpha) (1 + |\gamma''(\alpha)|) d\alpha ds.$$

It follows from (5.10) that

$$(5.12) \quad \int_{\mathcal{A}_k} |\tilde{K}_2(x)| \lesssim 2^{l-k},$$

and it follows from (5.11) that

$$(5.13) \quad \int_{\mathcal{A}_k} |\tilde{K}_1(x)| \lesssim 2^{l-k},$$

and (5.12) and (5.13) imply (5.5).

Now, (5.10) implies that for  $0 < \delta < C$  we have that  $|\tilde{K}_2 \cdot \chi_{|t^A \cdot| \geq 2^{5M \cdot l}}|$  is bounded above by a radial, decreasing function with  $L^1$  norm  $\lesssim 1$ . It follows that there is a sequence  $\{a_n\}$  with  $a_n \geq 0$  and  $\sum_{n=0}^{\infty} a_n \lesssim 1$  such that for  $0 < \delta < C$ ,

$$(5.14) \quad \left\| \sup_{t \in (0, \infty)} |(\chi_{|t^A \cdot| \geq 2^{5M \cdot l}} \cdot \det(t^A) \tilde{K}_2(t^A \cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| \sum_{n=0}^{\infty} a_n M_{2^{-n}, 1} f \right\| \lesssim \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)},$$

where we have applied Proposition 3.1.

We now prove a similar estimate for  $\tilde{K}_2$ . Observe that (5.11) implies that if  $0 < \delta < C$ ,

$$\sup_{t \in (0, \infty)} |(\chi_{|t^A \cdot| \geq 2^{5M \cdot l}} \cdot \det(t^A) \tilde{K}_1(t^A \cdot)) * f(x)| \lesssim \int \int \sum_{n=0}^{\infty} 2^{-n/4} M_{2^{n/4}, 2^{n/4}} f(x) |h_l''(x)| \zeta_0(\alpha) (1 + |\gamma''(\alpha)|) d\alpha ds,$$

and hence by Proposition 3.1,

$$(5.15) \quad \left\| \sup_{t \in (0, \infty)} |(\chi_{|t^A \cdot| \geq 2^{5M \cdot l}} \cdot \det(t^A) \tilde{K}_1(t^A \cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}.$$

Together (5.14) and (5.15) prove the result with  $\overline{M}$  replaced by  $M_1$ .

It remains to prove the result with  $\overline{M}$  replaced by  $M_2$ . Observe that

$$\begin{aligned} \sigma_j(\xi) &= \phi_0(2^{-2M})\alpha_j(\xi)(1 - \alpha_j(\xi - 2^{M+10}|I_j|(1, \gamma'_0(i_j)))) \\ &\quad \times (1 - \alpha_{j+1}(\xi))(\alpha_{j+1}(\xi + 2^{M+10}|I_j|(1, \gamma'_0(b_{j+1})))) \cdot \tilde{m}_j(\xi), \end{aligned}$$

where

$$\tilde{m}_j(\xi) = \Psi_0(\xi)\psi_0(\rho(\xi))\nu_j(((1/\rho(\xi))^A\xi)_1),$$

for some  $C^\infty$  function  $\nu_j$  supported in an interval  $I_j^*$  of width  $10|I_j|$  satisfying

$$|D^i\nu_j| \lesssim |I_j|^{-i}$$

for every integer  $i \geq 0$ . The kernel of the multiplier  $\tilde{m}_j$  can be easily written as an integral in homogeneous coordinates. If we can prove that for every annulus  $\mathcal{A}_k$  with  $k \geq 0$ ,

$$(5.16) \quad \int_{\mathcal{A}_k} |\mathcal{F}^{-1}[\tilde{m}_j(\cdot)](x)| dx \lesssim l,$$

then it would follow that the desired result reduces to proving the result of the proposition with  $\overline{M}f(x)$  replaced by

$$(5.17) \quad \sup_{t \in (0, \infty)} |(\chi_{|t^A| \leq 2^{20M}l} \cdot \mathcal{F}^{-1}[\tilde{m}_j(t^{-A}\cdot)]) * f(x)|.$$

We now proceed to prove (5.16). As before, let  $\eta$  be smooth and supported in  $[-\epsilon, \epsilon]$ , where  $\epsilon$  is given by (5.9). Also, as before let  $\phi \in C^\infty([-1, 1])$  be nonnegative and identically 1 on  $[-1/2, 1/2]$ , and for  $n \in \mathbb{Z}$  set  $\phi_n(\cdot) = \phi(2^{-n-1}\cdot) - \phi(2^{-n}\cdot)$ . Define

$$(5.18) \quad \Phi_0(x, s, \alpha) = \phi_0(|I_j| \langle x, s^A(1, \gamma'_0(\alpha)) \rangle) \eta\left(\frac{\langle x, s^A(1, \gamma'_0(\alpha)) \rangle}{|x|}\right)$$

$$(5.19) \quad \begin{aligned} \Phi_n(x, s, \alpha) &= (\phi_0(2^{-n-1}|I_j| \langle x, s^A(1, \gamma'_0(\alpha)) \rangle) \\ &\quad - \phi_0(2^{-n}|I_j| \langle x, s^A(1, \gamma'_0(\alpha)) \rangle)) \eta\left(\frac{\langle x, s^A(1, \gamma'_0(\alpha)) \rangle}{|x|}\right). \end{aligned}$$

We decompose the kernel as

$$(5.20) \quad \mathcal{F}^{-1}[\tilde{m}_j(\cdot)](x) = \frac{1}{(2\pi)^2} [\tilde{K}_j(x) + \sum_{n \geq 0} K_{j,n}(x)],$$

where

$$\begin{aligned} K_{j,n}(x) &= \int \nu_j(\alpha) \int h_l(s) \Phi_n(x, s, \alpha) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} \\ &\quad \times \langle s^A(1, \gamma'_0(\alpha)), s^{-1}As^A(\alpha, \gamma_0(\alpha)) \rangle ds d\alpha \end{aligned}$$

and

$$\begin{aligned} \tilde{K}_j(x) = & \int \nu_j(\alpha) \left(1 - \eta\left(\frac{\langle x, s^A(1, \gamma'_0(\alpha)) \rangle}{|x|}\right)\right) \int h_l(s) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} \\ & \times \langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle ds d\alpha. \end{aligned}$$

Note that the sum in (5.20) has only  $\lesssim \log(1 + |I_j||x|)$  terms, since  $K_{j,n}(x) = 0$  if  $2^{n-10}|I_j|^{-1} \geq \epsilon|x|$ . In particular if  $x \in \mathcal{A}_k \cap \text{supp}(K_{j,n})$  then  $2^n \ll 2^k |I_j|$ .

For  $K_{j,0}$ , we simply estimate  $\int_{\mathbb{R}^2} |K_{j,0}(x)| dx$ . For a given  $(\alpha, s)$ , we introduce coordinates

$$(5.21) \quad (u_1, u_2) \mapsto \xi(u_1, u_2) = u_1 s^A(1, \gamma'_0(\alpha)) + u_2 s^{-1} A s^A(1, \gamma'_0(\alpha)).$$

The Jacobian of the map (5.21) is  $\approx 1$ . Integrating by parts three times in  $s$  yields

$$(5.22) \quad |K_{j,0}(x)| \lesssim \int_{s: |s-1| \approx 2^{-l}} \int_{\substack{\alpha \in I_j^* \\ \langle x, s^A(1, \gamma'_0(\alpha)) \rangle \leq (2|I_j|)^{-1}}} (1 + 2^{-l} |\langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle|)^{-3} ds d\alpha,$$

and thus using the change of coordinates (5.21)

$$(5.23) \quad \int_{\mathbb{R}^2} |K_{j,0}(x)| dx \lesssim \int_{I_j^*} \int \int_{|u_1| \leq (2|I_j|)^{-1}} 2^{-l} (1 + 2^{-l} |u_2|)^{-3} du_1 du_2 d\alpha \lesssim 1.$$

For  $n > 0$ , we integrate by parts with respect to  $\alpha$  once and then with respect to  $s$  twice, which yields

$$K_{j,n}(x) = \int \int h_l(s) g_n(x, s, \alpha) e^{i\langle x, s^A(\alpha, \gamma_0(\alpha)) \rangle} d\alpha ds,$$

where

$$\begin{aligned} g_n(x, s, \alpha) = & -\frac{d}{ds} \left( \frac{1}{\langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle} \frac{d}{ds} \left( \frac{1}{\langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle} \right. \right. \\ & \left. \left. \frac{d}{d\alpha} \left( \frac{1}{\langle x, s^A(1, \gamma'_0(\alpha)) \rangle} \nu_j(\alpha) \Phi_n(x, s, \alpha) \langle s^A(1, \gamma'_0(\alpha)), s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle \right) \right) \right). \end{aligned}$$

On the support of  $h_l(s)$  we have

$$|g_n(x, s, \alpha)| \lesssim \frac{(1 + |x| 2^{-n} |I_j|) |\gamma_0''(\alpha)| + |I_j|^{-1}}{2^{-2l} |\langle x, s^{-1} A s^A(1, \gamma'_0(\alpha)) \rangle|^2 |\langle x, s^A(1, \gamma'_0(\alpha)) \rangle|},$$

and so

$$(5.24) \quad |K_{j,n}(x)| \lesssim \int_{s: |s-1| \approx 2^{-l}} \int_{\substack{\alpha \in I_j^* \\ |\langle x, s^A(1, \gamma'_0(\alpha)) \rangle| \\ \approx 2^n |I_j|^{-1}}} \frac{(1 + |x| 2^{-n} |I_j|) |\gamma''_0(\alpha)| + |I_j|^{-1}}{|\langle x, s^A(1, \gamma'_0(\alpha)) \rangle|} \frac{1}{(1 + 2^{-l} |\langle x, s^{-1} A s^A(1, \gamma'_0(\alpha)) \rangle|)^2} d\alpha ds.$$

Using the change of coordinates (5.21), it follows that

$$\begin{aligned} \int_{\mathcal{A}_k} |K_{j,n}(x)| dx &\lesssim \int_{s: |s-1| \approx 2^{-l}} 2^l \int_{\alpha \in I_j^*} ((1 + 2^{k-n} |I_j|) |\gamma''_0(\alpha)| + |I_j|^{-1}) \\ &\quad \times \int_{\substack{u_1 \approx 2^n |I_j|^{-1} \\ |u| \approx 2^k}} |u_1|^{-1} \frac{2^{-l}}{(1 + 2^{-l} |u_1|)^2} du d\alpha \\ &\lesssim \int_{I_j^*} (|\gamma''_0(\alpha)| + 2^{k-n} |I_j| |\gamma''_0(\alpha)| + |I_j|^{-1}) d\alpha. \end{aligned}$$

By (4.2) we have  $\int_{I_j^*} |I_j| |\gamma''_0(\alpha)| d\alpha \leq 2^{-l}$ , and so

$$\int_{\mathcal{A}_k} |K_{j,n}(x)| dx \lesssim \min\{2^{k-l}, 2^{l-k}\} (2^{k-l-n} + 1).$$

Since  $K_{j,n}$  is identically 0 on  $\mathcal{A}_k$  if  $n \geq k$ , summing in  $n$  and also using (5.23) yields

$$(5.25) \quad \sum_{n \geq 0} \int_{\mathcal{A}_k} |K_{j,n}(x)| dx \lesssim k 2^{-|k-l|}.$$

Now we estimate  $\tilde{K}_j$ . Integrating by parts once in  $\alpha$  and then once in  $s$  yields

$$(5.26) \quad |\tilde{K}_j(x)| \lesssim \int_{s: |s-1| \approx 2^{-l}} \int_{I_j^*} \frac{(|I_j|^{-1} + |\gamma''_0(\alpha)|)}{|x| (1 + 2^{-l} |\langle x, s^{-1} A s^A(\alpha, \gamma_0(\alpha)) \rangle|)} d\alpha ds,$$

and so using the change of coordinates (5.21) we get

$$(5.27) \quad \int_{\mathcal{A}_k} |\tilde{K}_j(x)| dx \lesssim 1.$$

Combining (5.25) and (5.27) gives (5.16). We now proceed to examine

$$\sup_{t \in (0, \infty)} |(\chi_{|t^A \cdot| \leq 2^{20M \cdot l}} \cdot \mathcal{F}^{-1}[\tilde{m}_j(t^{-A} \cdot)]) * f(x)|.$$

By (5.22), for  $0 < \delta < C$  we have

$$\sup_{t \in (0, \infty)} |(\chi_{|t^A| \leq 2^{20M \cdot l}} \cdot K_{j,0}(t^A \cdot)) * f(x)| \lesssim \int_{s: |s-1| \approx 2^{-l}} 2^l \int_{\alpha \in |I_j|^*} |I_j|^{-1} \sum_{n=0}^{C_M \cdot l} M_{|I_j|^{-1}, 2^{l+n/3}|I_j|} f(x) d\alpha ds,$$

and hence by Proposition 3.1,

$$(5.28) \quad \left\| \sup_{t \in (0, \infty)} |(\chi_{|t^A| \leq 2^{20M \cdot l}} \cdot K_{j,0}(t^A \cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim_{\epsilon} \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}.$$

Similarly examining (5.24) and (5.26) leads to

$$(5.29) \quad \left\| \sup_{t \in (0, \infty)} |(\chi_{|t^A| \leq 2^{20M \cdot l}} \cdot K_{j,n}(t^A \cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim_{\epsilon} \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}$$

for  $n > 0$  and

$$(5.30) \quad \left\| \sup_{t \in (0, \infty)} |(\chi_{|t^A| \leq 2^{20M \cdot l}} \cdot \tilde{K}_j(t^A \cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim_{\epsilon} \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}.$$

Combining (5.28), (5.29) and (5.30) proves the result with  $\overline{M}f(x)$  replaced by (5.17), and the proof of the proposition is complete.  $\square$

Finally, we note that the proof of Proposition 5.1 implies the following  $L^1$  kernel estimate.

**Proposition 5.2.** *There exists a constant  $C = C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A))$  such that for  $0 < \delta < C$ , for every  $\epsilon > 0$  and for every quadruple  $(i, j, m, n)$ ,*

$$\left\| \sup_{t \approx 2^n} |\psi_t * K_{i,j,m,n}| \right\|_{L^1(\mathbb{R}^2)} \lesssim_{\epsilon, M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)} 1.$$

The above estimate without the supremum follows immediately from the proof of Proposition 5.1. We then simply note that all  $L^1$  kernel estimates in the proof of Proposition 5.1 follow from pointwise estimates, which still hold uniformly in  $t$  when the kernel is convolved with  $\psi_t$ .

## 6. LITTLEWOOD-PALEY INEQUALITIES

The goal of this section is to prove the following proposition, which is an analog of Proposition 4 from [2]. As noted in the introduction, the presence of nonisotropic dilations requires a more complicated application of square function estimates than those used in [2], where Proposition 4 is proved



by iteratively applying square function estimates with respect to Fourier projections to parallel strips in  $\mathbb{R}^2$ .

**Proposition 6.1.** *Let  $\epsilon > 0$ . There is  $C = C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A), \epsilon) > 0$  such that if  $0 < \delta < C$ , then the following holds. Let  $\{\sigma_{i,j,m,n}\}$  be the partition of unity constructed in section 5 for the given value of  $\delta$ . There are smooth functions  $\{\phi_{i,j,m,n}\}$  such that  $\phi_{i,j,m,n}$  is identically 1 on the support of  $\sigma_{i,j,m,n}$  and so that if we define  $\tilde{P}_{i,j,m,n}$  to be the convolution operator whose multiplier is  $\phi_{i,j,m,n}$ , then*

$$(6.1) \quad \left\| \left( \sum_{i,j,m,n} |\tilde{P}_{i,j,m,n} f|^2 \right)^{1/2} \right\|_4 \lesssim_{\epsilon} \delta^{-\epsilon} \|f\|_4.$$

To prove Proposition 6.1, we will need the following lemma, which was originally due to Carleson. A proof can be found in [13] (Lemma 4.4). We state the lemma in full generality, although we will only need the special case  $d = 2$ .

**Lemma 6.2.** *Let  $A$  be an invertible linear transformation on  $\mathbb{R}^d$  and  $A^t$  its transpose. Suppose that  $\{m_k\}_{k \in \mathbb{N}}$  are bounded, measurable functions on  $\mathbb{R}^d$  with disjoint supports. Let  $w$  be a bounded, measurable function on  $\mathbb{R}^d$ . Then for  $s \geq 0$  and  $f \in \mathcal{S}(\mathbb{R}^d)$ ,*

$$\begin{aligned} & \int \sum_k |\mathcal{F}^{-1}[m_k(A^t \cdot) \hat{f}](x)|^2 w(x) dx \\ & \leq C \sup_k \|m_k\|_{L_s^2(\mathbb{R}^d)}^2 \int \int \frac{\det(A^{-1})}{(1 + |A^{-1}y|^s)^2} |f(x-y)|^2 dy w(x) dx. \end{aligned}$$

We state an immediate corollary of this lemma, which we will apply repeatedly in the proof of Proposition 6.1.

**Corollary 6.3.** *Suppose that  $\{m_k\}_{k \in \mathbb{Z}}$  are disjoint translates of a smooth compactly supported function adapted to the unit cube in  $\mathbb{R}^2$ , with the distance between the supports of the  $m_k$  at least  $O(1)$ . Let  $R_\theta$  be the matrix of rotation by  $\theta$  degrees, and for  $n \in \mathbb{Z}$  put  $A_{n,\theta} = ((2^n)^A R_\theta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda N \end{pmatrix})^t$ . Then for any  $n, \theta$  and for any  $s > 0$ ,*

$$\int \sum_k |\mathcal{F}^{-1}[m_k(A_{n,\theta}^t \cdot) \hat{f}](x)|^2 w(x) dx \leq C \int |f(x)|^2 \mathcal{M}_{\lambda,N} w(x) dx,$$

where  $\mathcal{M}_{\lambda,N} := \sum_{i=0}^{\infty} 2^{-i} M_{2^i \lambda, N}$ .

*Proof of Proposition 6.1.* Without loss of generality, we may restrict the sum in (6.1) to  $i = 0$ , and so in what follows we will assume  $i = 0$  and drop the  $i$ -index. Also, in what follows we will say a collection  $\mathcal{R}$  of subsets of  $\mathbb{R}^2$  is

*almost disjoint* if there is a constant  $C = C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)) > 0$  such that every point of  $\mathbb{R}^2$  is contained in at most  $C$  elements of  $\mathcal{R}$ .

The main difficulty here introduced by nonisotropic dilations is that unlike the isotropic case, the orbits  $\{t^A \xi : t > 0\}$  need not be straight lines, and thus for fixed  $j$  the supports of the  $\sigma_{j,m,0}$  may only be approximated by rectangles with axes whose directions change as  $m$  varies. To deal with this difficulty, we group the supports of the  $\sigma_{j,m,0}$  into nested subcollections each of which can be approximated by rectangles with long axes in a single direction, and iteratively apply Corollary 6.3.

Note that since  $|\delta| \lesssim |I_j| \lesssim 1$ , there are  $\lesssim \log(1/\delta)$  dyadic intervals  $[2^a, 2^{a+1}]$  with  $a \leq 0$  and  $a \in \mathbb{Z}$  such that  $2^a \leq |I_j| \leq 2^{a+1}$  for some  $j$ , and so if we let  $\mathfrak{J}_a = \{j : |I_j| \in [2^a, 2^{a+1}]\}$ , we may restrict the sum in  $j$  in (6.1) to  $\mathfrak{J}_a$  for a single fixed value of  $a$ , as long as all our estimates are uniform in  $a$ . By incurring a factor of  $\delta^{-\epsilon}$ , we may assume that  $2^a \leq \delta^{-\epsilon}$ .

Having fixed  $a$ , we are now ready to construct for each fixed  $j$  our nested subcollections of indices  $m$ . The idea is that for a fixed  $j$  and a fixed  $m$ , the support of  $\sigma_{j,m,0}$  is essentially a  $2^a \times \delta$  rectangle, and the support of  $\sigma_{j,m',0}$  for  $m'$  for  $|m' - m| \lesssim 2^{-a}$  is contained in a  $2^a \times \delta$  rectangle whose direction differs by at most  $\lesssim 2^{-a}\delta$ . Thus the supports of the functions  $\{\sigma_{j,m',0}\}_{|m-m'| \lesssim 2^{-a}}$  are contained in almost disjoint parallel strips of width  $\approx \delta$ . For such a collection of rectangles, Corollary 6.3 may be applied. The union of such rectangles is essentially a  $2^a \times 2^{-a}\delta$  rectangle. We now iterate this process, grouping together successive  $2^a \times 2^{-a}\delta$  rectangles whose direction does not change too much to obtain a rectangle of smaller eccentricity. We continue this process until we obtain a  $2^a \times 2^a$  square, and then we may apply Corollary 6.3.

The nested subcollection of indices  $m$  will be constructed “backwards” with respect to the process described in the previous paragraph. The number of stages required by the process is  $N$ , where  $N$  is the least integer such that  $2^{aN} \leq \delta$ . For each  $1 \leq k \leq N$ , we will define a collection of indices  $m$  denoted by  $\mathfrak{M}_{i_1, \dots, i_k}$ , so that  $\mathfrak{M}_{i_1, \dots, i_{k+1}} \subset \mathfrak{M}_{i_1, \dots, i_k}$  and so that  $\mathfrak{M}_{i_1, \dots, i_N}$  contains at most one element. For each  $(i_1, \dots, i_k) \in \mathbb{Z}^k$ , inductively define

$$\mathfrak{M}_{i_1} = \{m : i_1 \lfloor 2^a \delta^{-1} \rfloor \leq m < (i_1 + 1) \lfloor 2^a \delta^{-1} \rfloor\},$$

$$\begin{aligned} \mathfrak{M}_{i_1, \dots, i_k} &= \mathfrak{M}_{i_1, \dots, i_{k-1}} \cap \{m : \sum_{1 \leq l \leq k} i_l \lfloor 2^{al} \delta^{-1} \rfloor \leq m \\ &\leq \sum_{1 \leq l \leq k} i_l \lfloor 2^{al} \delta^{-1} \rfloor + \lfloor 2^{ak} \delta^{-1} \rfloor\}. \end{aligned}$$

Then for every  $N$ -tuple  $(i_1, \dots, i_N)$ ,  $\mathfrak{M}_{i_1, \dots, i_N}$  contains at most one element.

Now let  $C = C(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2), \Theta(\Omega, A)) > 0$  be sufficiently large. There is a collection  $\{Q_{j,i_1}\}$  of almost-disjoint cubes of sidelength  $C2^a$  such that if  $m \in \mathfrak{M}_{i_1, \dots, i_N}$  then the support of  $\sigma_{j,m,0}$  is contained in  $Q_{j,i_1}$ . Since  $\sup_{\rho(\xi) \leq 8} |\nabla \rho(\xi)| \lesssim 1$ , there is a constant  $C > 0$  such that for every  $j, i_1$

we may cover  $Q_{j,i_1}$  with almost disjoint parallel rectangles  $R_{j,i_1,i_2}$  of width  $C2^{2a}$  and length 1 so that for every  $i_2$ ,

$$\bigcup_{m \in \mathfrak{M}_{i_1,i_2}} \text{supp}(\sigma_{j,m,0}) \subset \bigcup_r (R_{j,i_1,i_2} \cap Q_{j,i_1}).$$

Repeating this process, for every  $2 \leq k \leq N$  and every  $k$ -tuple  $(i_1, \dots, i_{k-1})$  we obtain almost disjoint parallel rectangles  $R_{j,i_1,\dots,i_k}$  of width  $C2^{ka}$  and length 1 so that for every  $i_k$ ,

$$\bigcup_{m \in \mathfrak{M}_{i_1,\dots,i_k}} \text{supp}(\sigma_{j,m,0}) \subset (R_{j,i_1,\dots,i_k} \cap \dots \cap R_{j,i_1,i_2} \cap Q_{j,i_1}).$$

As noted previously, in the case  $k = N$ ,  $\bigcup_{m \in \mathfrak{M}_{i_1,\dots,i_N}} \text{supp}(\sigma_{j,m,0})$  contains at most one element.

Let  $\phi : \mathbb{R}^2 \rightarrow [0, 1]$  be a smooth function that is identically 1 on the unit cube centered at the origin and supported in its double dilate. If  $R$  is any nonisotropic dilate of a rectangle, let  $L_R$  be the affine transformation taking  $R$  to the unit cube centered at the origin. It follows that if  $m \in \mathfrak{M}_{i_1,\dots,i_N}$ , then

$$\text{supp}(\sigma_{j,m,0}) \subset \{x : \phi(L_{Q_{j,i_1}}x) \prod_{l=2}^N \phi(L_{R_{j,i_1,\dots,i_l}}x) = 1\},$$

and so

$$\text{supp}(\sigma_{j,m,n}) \subset \{x : \phi(L_{Q_{j,i_1}}(2^{-n})^A x) \prod_{l=2}^N \phi(L_{R_{j,i_1,\dots,i_l}}(2^{-n})^A x) = 1\}.$$

Now for each  $j, i_1$  let  $\psi_{Q_{j,i_1}}$  be a smooth function supported in  $4Q_{j,i_1}$  and identically 1 on  $Q_{j,i_1}$ , so that for each  $j$ ,

$$\psi_{Q_{j,i_1}}(x) = 1, \quad x \in \bigcup_{m \in \mathfrak{M}_{i_1}} \text{supp}(\sigma_{j,m,0}).$$

For each  $(j, m, n)$ , let  $(i_1, \dots, i_N)$  be the unique  $N$ -tuple such that  $m \in \mathfrak{M}_{i_1,\dots,i_N}$ , and let

$$\phi_{j,m,n} = \psi_{Q_{j,i_1,r}}((2^{-n})^A x) \prod_{l=2}^N \phi(L_{R_{j,i_1,\dots,i_l,r}}(2^{-n})^A x)$$

Let  $\tilde{P}_{j,m,n}$  denote the convolution operator with multiplier  $\phi_{j,m,n}$ . Let  $\phi : \mathbb{R} \rightarrow [0, 1]$  be a smooth function supported in  $(1/4, 4)$  that is identically 1 on  $[1/2, 2]$ , and let  $P_n$  denote the convolution operator with multiplier  $\phi(2^{-n}\rho(\cdot))$ . Given an  $N$ -tuple of indices  $(i_1, \dots, i_N)$ , let  $m(i_1, \dots, i_N)$  denote the unique value of  $m$  such that  $m \in \mathfrak{M}_{i_1,\dots,i_N}$ , and let  $m(i_1, \dots, i_N)$  be undefined otherwise. Let  $S_{j,i_1}$  denote the convolution operator with multiplier

$$\psi_{Q_{j,i_1,r}}((2^{-n})^A \cdot).$$

For  $2 \leq k \leq N$ , let  $S_{j,i_1,\dots,i_k,n}$  denote the convolution operator with multiplier

$$\psi_{Q_{j,i_1}}((2^{-n})^A L_{j,i_1,r \cdot}) \phi(L_{R_{j,i_1,\dots,i_k}}(2^{-n})^A \cdot).$$

Then since each index  $m$  is contained in at most one  $N$ -tuple  $(i_1, \dots, i_N)$ , it follows that

$$\begin{aligned} \int \sum_{j,m,n} |\tilde{P}_{j,m,n} f(x)|^2 w(x) dx &= \\ \int \sum_n \sum_j \sum_{(i_1,\dots,i_N)} |S_{j,i_1,\dots,i_N}(\dots(S_{j,i_1}(P_n f(x)))\dots)|^2 w(x) dx. \end{aligned}$$

Repeatedly applying Corollary 6.3, we have

$$\begin{aligned} (6.2) \quad & \int \sum_{j,m,n} |\tilde{P}_{j,m,n} f(x)|^2 w(x) dx \\ & \lesssim_\epsilon \int \sum_n \sum_j \sum_{i_1,\dots,i_{N-1}} |S_{j,i_1,\dots,i_{N-1}}(\dots(S_{j,i_1}(P_n f(x)))\dots)|^2 \\ & \quad \times \mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w(x) dx \\ & \lesssim_\epsilon \int \sum_n \sum_j \sum_{i_1} |S_{j,i_1}(P_n f(x))|^2 \mathcal{M}_{1,(2^{2a}\delta^{2\epsilon})^{-1}}(\dots \\ & \quad (\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w(x)) \dots) dx \\ & \lesssim_\epsilon \int \sum_n |P_n f(x)|^2 \mathcal{M}_{2-a\delta^{-\epsilon},1}(\mathcal{M}_{1,(2^{2a}\delta^{2\epsilon})^{-1}}(\dots \\ & \quad (\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w(x)) \dots)) dx \\ & \lesssim_\epsilon \delta^{-\epsilon} \left\| \left( \sum_n |P_n f|^2 \right)^{1/2} \right\|_4^2 \\ & \quad \times \left\| \mathcal{M}_{2-a\delta^{0\epsilon},1}(\mathcal{M}_{1,(2^{2a}\delta^\epsilon)^{-1}}(\dots(\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w) \dots)) \right\|_2. \end{aligned}$$

By Proposition 3.1, we have

$$(6.3) \quad \left\| \mathcal{M}_{2-a\delta^{0\epsilon},1}(\mathcal{M}_{1,(2^{2a}\delta^\epsilon)^{-1}}(\dots(\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w) \dots)) \right\|_2 \lesssim_\epsilon \delta^{-\epsilon} \|w\|_2.$$

Since the operator  $f \mapsto \left( \sum_n |P_n f|^2 \right)^{1/2}$  corresponds to a vector-valued singular integral on the space of homogeneous type given by nonisotropic balls and Lebesgue measure with all associated constants  $\lesssim 1$ , we have

$$(6.4) \quad \left\| \left( \sum_n |P_n f|^2 \right)^{1/2} \right\|_4 \lesssim \|f\|_4.$$

Combining (6.2), (6.3) and (6.4), we have

$$(6.5) \quad \int \sum_{j,m,n} |\tilde{P}_{j,m,n} f(x)|^2 w(x) dx \lesssim_{\epsilon} \delta^{-\epsilon} \|f\|_4^2 \|w\|_2,$$

and the result follows by duality.  $\square$

## 7. PROOF OF THE MAIN THEOREM

In this section, we combine the ingredients developed in previous sections to prove Proposition 1.3. The argument will closely follow [2]. As noted previously, we only need prove Proposition 1.3 in the case that  $\Omega$  has smooth boundary, with a constant depending only on  $M, \text{Re}(\lambda_1), \text{Re}(\lambda_2), \Theta(\Omega, A), \epsilon$ .

*Proof of Propostion 1.3.* Let  $Sf(x) = \left( \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}$ . Let  $\mathfrak{S}$  be the nonisotropic sector bounded by the orbits  $\{t^A \xi : t > 0\}$  and  $\{t^A \xi' : t > 0\}$ , where  $\xi = (\xi_1, \xi_2)$  is the unique point in  $\partial\Omega$  with  $\xi_1 = -1/8$  and  $\xi_2 > 0$  and  $\xi' = (\xi'_1, \xi'_2)$  is the unique point in  $\partial\Omega$  with  $\xi_1 = 1/8$  and  $\xi_2 > 0$ . Assume without loss of generality that  $\hat{f}$  is supported in  $\mathfrak{S}$ . By incurring a factor of  $\log(1/\delta^{N(M, \text{Re}(\lambda_1), \text{Re}(\lambda_2))})$  we may restrict the domain of integration in  $t$  to the set

$$E = \bigcup_{\substack{n \equiv 0 \pmod{\log(1/\delta^{N(M, \text{Re}(\lambda_1), \text{Re}(\lambda_2))})}}} (2^n, 2^{n+1}],$$

where  $N(M, \text{Re}(\lambda_1), \text{Re}(\lambda_2))$  is as in Proposition 4.2. Now, if  $u, t \in E$  with  $u < t$ , then either  $u, t$  are contained in the same dyadic interval and  $u/t > 1/2$ , or  $u, t$  are contained in distinct dyadic intervals, and  $u/t < 1/\delta^{N(M, \text{Re}(\lambda_1), \text{Re}(\lambda_2))}$ . Using Plancherel, we have

$$\begin{aligned} \|Sf\|_4^4 &= \int \left| \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right|^2 dx = \\ &= \int \int_0^\infty \int_0^\infty |\psi_t * f(x)|^2 |\psi_u * f(x)|^2 \frac{dt}{t} \frac{du}{u} dx = \\ &= \int_0^\infty \int_0^\infty \int |(\phi(\frac{\rho(\cdot)}{t}) \hat{f}(\cdot)) * (\phi(\frac{\rho(\cdot)}{u}) \hat{f}(\cdot))(\xi)|^2 d\xi \frac{dt}{t} \frac{du}{u}. \end{aligned}$$

Restricting the integration in  $t$  and  $u$  to  $E$ , we have

$$\begin{aligned} \int_E \int_E \int |(\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) * (\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2 d\xi \frac{dt}{t} \frac{du}{u} \lesssim \\ \left( \int \int_{1/2 < t/u < 2} + \int \int_{u/t < \delta^{-N(M, \operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))}} \right) \int |(\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) \\ * (\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2 d\xi. \end{aligned}$$

Using Propositions 4.1 and 4.2, for every  $\epsilon > 0$  we can essentially bound this by

$$\begin{aligned} \delta^{-\epsilon} \left( \int_0^\infty \int_0^\infty \int \sum_{\substack{j,m,n \\ j',m',n'}} |(\sigma_{0,j,m,n}(\cdot)\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) \\ * (\sigma_{0,j',m',n'}(\cdot)\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2 d\xi \frac{dt}{t} \frac{du}{u} \right. \\ \left. + \int_0^\infty \int_0^\infty \int \sum_{j',m',n'} |(\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) * (\sigma_{0,j',m',n'}(\cdot)\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2 d\xi \frac{dt}{t} \frac{du}{u} \right). \end{aligned}$$

Let

$$Tf(x) = \left( \int_0^\infty \sum_{j,m,n} |\mathcal{F}[\sigma_{0,j,m,n}(\cdot)\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)](x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Then the above implies that

$$\|Sf\|_4^4 \lesssim_\epsilon \delta^{-\epsilon} (\|Tf\|_4^4 + \|Sf\|_4^2 \|Tf\|_4^2),$$

which implies

$$(7.1) \quad \|Sf\|_4 \lesssim_\epsilon \delta^{-\epsilon} \|Tf\|_4.$$

Using Proposition 4.3, we have

$$\begin{aligned} \|Tf\|_4 &= \left( \int \left| \int_0^\infty \sum_{j,m,n} |\mathcal{F}[\sigma_{0,j,m,n}(\cdot)\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)](x)|^2 \frac{dt}{t} \right| dx \right)^{1/4} \\ &= \left( \int \left| \sum_{j,m,n} \int_0^\infty |((\psi_t * K_{0,j,m,n}) * (\tilde{P}_{0,j,m,n}f))(x)|^2 \frac{dt}{t} \right| dx \right)^{1/4} \\ &\lesssim \delta^{1/2} \left( \int \left| \sum_{j,m,n} \sup_{t \approx 2^n} |(\psi_t * K_{0,j,m,n}) * (\tilde{P}_{0,j,m,n}f)(x)|^2 \right| dx \right)^{1/4} \\ &= \delta^{1/2} \left\| \left( \sum_{j,m,n} \sup_{t \approx 2^n} |(\psi_t * K_{0,j,m,n}) * (\tilde{P}_{0,j,m,n}f)(x)|^2 \right)^{1/2} \right\|_4. \end{aligned}$$

Now let  $\omega \in \mathcal{S}(\mathbb{R}^2)$  with  $\|\omega\|_{L^2(\mathbb{R}^2)} = 1$ . We have

$$\begin{aligned}
& \int \sum_{j,m,n} \sup_{t \approx 2^n} |\psi_t * K_{0,j,m,n} * \tilde{P}_{0,j,m,n} f(x)|^2 \omega(x) dx \\
& \lesssim \int \sum_{j,m,n} \left\| \sup_{t \approx 2^n} |\psi_t * K_{0,j,m,n}| \right\|_1 |\tilde{P}_{0,j,m,n} f(x)|^2 \sup_{t \approx 2^n} |\psi_t * K_{0,j,m,n} * w(x)| dx \\
& \lesssim_\epsilon \delta^{-\epsilon} \int \sum_{j,m,n} |\tilde{P}_{0,j,m,n} f(x)|^2 \overline{M} \omega(x) dx \\
& \lesssim_\epsilon \left\| \left( \sum_{j,m,n} |\tilde{P}_{0,j,m,n} f(x)|^2 \right)^{1/2} \right\|_4^2 \|\overline{M} w\|_2 \lesssim_\epsilon \delta^{-\epsilon} \|f\|_4^2,
\end{aligned}$$

where in the second inequality we have used Proposition 5.2 and in the last inequality we have used Propositions 5.1 and 6.1. Using (7.1) and taking the supremum over all such weights  $\omega$ , we have

$$\|Sf\|_4 \lesssim_\epsilon \delta^{-\epsilon} \|Tf\|_4 \lesssim_\epsilon \delta^{1/2-\epsilon} \|f\|_4.$$

□

*Proof that Proposition 1.3 implies Theorem 1.1.* Let  $C$  be as in the statement of Proposition 1.3. We will now decompose the Bochner-Riesz multipliers in a standard fashion. Let  $\phi_0 : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function identically 1 on  $[-1, 1]$  and supported in  $[-2, 2]$  so that  $\phi_0(|\cdot|)$  is a radial, decreasing function on  $\mathbb{R}^2$ . It is easy to see that we can find smooth functions  $\phi_1 : \mathbb{R} \rightarrow \mathbb{R}$  and  $\phi_2 : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the following:

- (1) For each  $k \geq 0$ ,

$$|D^k \phi_1(x)| \lesssim_k 1,$$

$$|D^k \phi_2(x)| \lesssim_k 1,$$

- (2) There is a constant  $C' > 0$  such that  $\phi_1$  is supported in  $[C', 1]$ ,  
(3) We can write

$$\begin{aligned}
(1 - \rho(\xi))^\lambda &= \phi_0(2^{2M}|\xi|) + (\phi_0(2^{-2M}|\xi|) - \phi_0(2^{2M}|\xi|))\phi_1(\rho(\xi)) \\
&\quad + \sum_{k=\lceil \log(C) \rceil}^{\infty} 2^{-k\lambda} \phi_2(2^k(1 - \rho(\xi))).
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned} \|G^\lambda f\|_4 &\lesssim \left\| \left( \int_0^\infty \left| \mathcal{F}^{-1}[\phi_0(2^{2M}t^{-1}|\cdot|)] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \\ &+ \left\| \left( \int_0^\infty \left| \mathcal{F}^{-1}[(\phi_0(2^{-2M}t^{-1}|\cdot|) - \phi_0(2^{2M}t^{-1}|\cdot|))\phi_1(t^{-1}\rho(\cdot))] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \\ &\quad + \sum_{k=1}^\infty 2^{-k\lambda} \left\| \left( \int_0^\infty \left| \mathcal{F}^{-1}[\phi_1(2^k(1 - t^{-1}\rho(\cdot)))] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4. \end{aligned}$$

The first term is clearly  $\lesssim \|f\|_4$ . By Proposition 1.3, the third term is also  $\lesssim \|f\|_4$  if  $\lambda > -1/2$ . By vector-valued singular integrals, the second term is bounded by

$$\left\| \left( \int_0^\infty \left| \mathcal{F}^{-1}[\phi_1(t^{-1}\rho(\cdot))] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4,$$

and it is straightforward to adapt the proof of Proposition 1.3 to show that this is  $\lesssim \|f\|_4$ .  $\square$

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